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GAS BUBBLES IN A LIQUID  
by Ignatz Reiter, Ph.D.  
Columbia University  
Division of War Research  
Section No. 3.1-20-127 7.15.41

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Division 6 - Section 6.1

ACOUSTIC PROPERTIES OF  
GAS BUBBLES IN A LIQUID

by

Lyman Spitzer, Jr.

Columbia University  
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Submitted for Columbia University  
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## PREFACE

Accumulating evidence indicates that the scattering and absorption of sound by small gas bubbles in water can constitute a serious difficulty in echo ranging or in listening. Work in connection with both the Wake and Reverberation Programs indicates that a study of the acoustic properties of small bubbles is necessary for a complete understanding of the transmission of sound in sea water. In addition, air bubbles are apparently the most efficient absorbers of underwater sound and are, therefore, of interest in all cases where an absorbing sound screen would be useful.

A considerable amount of research has been done on this subject, but the results have not hitherto been available in a simple comprehensive form. The present report, which is intended primarily for the use of research workers in underwater acoustics, attempts to summarize and bring together in one place all relevant information on the acoustic properties of gas bubbles.

The final results, which make possible in certain cases a prediction of the reflection, scattering, and absorption to be expected from a given distribution of bubbles, are summarized in the first few pages of the report. For many practical purposes a reading of this Summary will be sufficient. The remainder of the report, which may be regarded as an appendix, will be of interest to those concerned with the derivation and observational verification of the formulae given in the Summary.



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SUMMARY

Small gas bubbles in a liquid can be very effective scatterers and absorbers of sound, owing to the possibility of resonance between the sound field and the natural oscillations of the bubble. The radius  $R_T$  at which a bubble resonates to sound of frequency  $\nu$  is given by the equation

$$R_T = \frac{1}{2\pi\nu} \sqrt{\frac{3\gamma P_0 g}{\rho \alpha}} ;$$

$P_0$  is the hydrostatic pressure,  $\rho$  is the density of the liquid, and  $\gamma$  is the ratio of specific heats of the gas in the bubbles;  $\alpha$  is a quantity which increases from 1 to  $\gamma$  as  $R$  decreases, corresponding to the transition from adiabatic to isothermal conditions within the bubble. The value of  $\alpha$  is given from theory in terms of the density, specific heat, and heat conductivity of the gas, - see Section II. The quantity  $g$  is a correction factor which takes surface tension into account, and is given by the equation

$$g = 1 + \frac{2T}{R P_0} \left\{ 1 - \frac{\alpha}{3\gamma} \right\} ,$$

where  $T$  is the surface tension of the liquid-gas interface..

For an air bubble in water,  $\gamma$  is 1.4,  $\rho$  is 1.0, and the equation for  $R_T$  may be put in the more convenient form

$$R_r(\text{cm}) = \frac{0.33}{\nu(\text{kc})} \sqrt{\frac{gP_o(\text{atm})}{\alpha}},$$

where now the pressure is in atmospheres. This equation has been verified observationally within a few percent from 2 kc up to 30 kc, in which range both  $g$  and  $\alpha$  are essentially unity. Values of  $R_r$  for different  $\nu$  and  $P_o$  are given in the accompanying Table.

TABLE I.

Resonant Radius for Air Bubble in Water

Frequency:			1 kc	5 kc	20 kc	50 kc	500 kc
Pressure	1	surface	.33 cm	.065 cm	.016 cm	.0063 cm	.00062 cm
in							
Atmospheres	2	35 ft.	.47	.093	.023	.0091	.00093
and							
Correspond-	5	140 ft.	.73	.15	.037	.015	.0015
ing Depth							
of Water:	10	300 ft.	1.04	.21	.052	.021	.0022

The scattering and absorption produced by a single bubble may conveniently be expressed in terms of scattering and absorption "cross-sections". The scattering cross-section  $\sigma_s$  is defined so that the total energy of the scattered radiation is just equal to the amount of incident energy passing through an area  $\sigma_s$  placed perpendicular to the beam. Similarly the total energy absorbed may be represented by an absorption cross-section  $\sigma_a$  such that the total incident sound energy passing through an area  $\sigma_a$  per second is just equal to the energy absorbed per second by the bubble. The sum of  $\sigma_a$  and  $\sigma_s$  is called the extinction cross-section and is denoted by  $\sigma_e$ ; it represents the total energy

removed from the beam through scattering plus absorption.

The theory shows that for sound of frequency  $\nu$  incident upon a bubble of radius R

$$\sigma_s = \frac{4\pi R^2}{(\nu_r^2/\nu^2 - 1)^2 + \delta^2},$$

$$\sigma_e = \frac{4\pi R^2 \delta/a}{(\nu_r^2/\nu^2 - 1)^2 + \delta^2},$$

where  $\nu_r$  is the resonant frequency for the bubble. The quantity  $\delta$  is called a "damping constant", and is the sum of three terms, representing the damping effects of radiation, viscosity and heat conduction;  $a$  is the contribution of radiation damping to  $\delta$ , and at resonance is equal to

$$a = \sqrt{\frac{3\gamma P_0 g}{\rho c^3 a}} = 1.36 \times 10^{-2} \sqrt{\frac{g}{a}}$$

if  $\gamma$ ,  $\rho$ , and the sound velocity  $c$  are taken for air bubbles in water at a pressure of one atmosphere.

The theory indicates that the effect of viscosity may be neglected for bubbles greater than  $3 \times 10^{-4}$  cm in radius, corresponding to frequencies of resonance less than a megacycle. The theoretical determination of  $\delta$  is not to be trusted, however, since the available observations show values of  $\delta$  of about 0.27 for resonant bubbles at 24 kc, as compared with a theoretical value of 0.08.

When many bubbles are present, the absorption and scattering remove energy from the incident beam, giving rise to an attenuation which may be expressed as  $K_e$  db per yard. The attenuation of the sound beam which results from scattering alone may be expressed as  $K_s$  db per yard. The difference between  $K_e$  and  $K_s$  is  $K_a$ , that part of the total attenuation which arises from absorption alone. If an integration is carried out only over bubbles very near resonance,

$$K_e = \frac{1.4 \times 10^5}{R_r} u_r \text{ db/yard,}$$

$$K_s = 5.2 \times 10^5 u_r \text{ db/yard,}$$

where the resonant radius  $R_r$  is in centimeters. The quantity  $u_r$  is the volume of air occupied by resonant bubbles per cubic centimeter per unit interval of  $\log_e R$ . If the relative volume occupied by bubbles of different sizes does not change rapidly with  $R$ , then  $u_r$  is roughly the volume of all bubbles in a cubic centimeter with radii between  $R_r/2$  and  $3R_r/2$ . Since  $R_r$  is much less than a millimeter in the cases of practical interest,  $K_e$  is much greater than  $K_s$ . This corresponds to the fact that resonant bubbles at supersonic frequencies absorb considerably more energy than they scatter.

The quantity  $K_e$ , which represents the total attenuation of the initial beam in db per yard, may be observed directly.

The value of  $K_e$  given in the above formula is independent of the damping constant  $\delta$  and should give a reliable determination of the contribution which resonant bubbles make to the total attenuation. The numerical value of  $K_s$  given, however, depends on the observed values of  $\delta$  at resonance and is valid only for values of  $R_r$  less than 0.1 cm. The value obtained is independent of frequency but may be expected to be different at pressures other than one atmosphere.

Since  $K_s$  represents the energy removed from the main beam by scattering, its value may be used to determine the amount of radiation scattered. If  $I$  is the intensity of the incident radiation, each bubble will scatter an amount of energy  $\sigma_s I$  per second, and the total energy scattered from a unit volume will be  $n\sigma_s I$ , where  $n$  is the number of bubbles per unit volume. The San Diego group has denoted  $n\sigma_s$  by the symbol  $m$ . In the present notation

$$m = K_s \frac{\log_e 10}{10} = 1.2 \times 10^5 u_r \text{ yard}^{-1}.$$

The total energy scattered from a cubic yard will be  $mI$ , provided that  $K_e$  is less than 1 db so that  $I$  is uniform throughout the volume in question. This scattered sound will be of equal intensity in all directions.

When the scattering is being computed from a large volume, or when  $K_s$  is very great, the incident intensity  $I$  is differ-

ent in different regions, multiple scattering becomes important, and the computation of the scattered radiation is more complicated. If the attenuation  $K_e$  is much greater than  $K_s$ , however, the scattered radiation may be computed. For a layer of thickness  $X$  in which  $K_s$  and  $K_e$  are constant, and in which the total attenuation  $K_e X$  is large, almost all of the incident energy  $I$  will be either scattered or absorbed. A fraction  $K_s/K_e$  will be scattered, but only half of this will be scattered backwards, and even this half will be partly absorbed on its way back out of the layer. The result is that the radiation scattered directly backwards out of the layer may be computed as though a fraction  $K_s/4K_e$  of the energy  $I$  were scattered in all backward directions, i.e., over a hemisphere.

Bubbles other than those near resonance have a smaller acoustic effect than those near resonance, unless the number of resonant bubbles is relatively very small. Under some circumstances, bubbles above resonance may contribute to the scattering, while those below resonance may be important in absorbing sound. For bubbles whose radii are below resonance, but greater than  $0.1R_r$ , the total absorption will be  $1/20$  to  $1/40$  as great as that from the resonance peak if the total geometrical cross-section of all bubbles per radius interval per  $\text{cm}^3$  is roughly constant for the entire range of bubble sizes. On the other hand, for microscopic bubbles, with a radius below  $3 \times 10^{-4}$  cm, viscosity is important and  $K_a$ , the attenuation at 24 kc, in db per yard, is

$$K_a = 1.8 \times 10^4 \frac{u}{g^2} \text{ db/yard} ;$$

$u$  is the total volume of such bubbles per  $\text{cm}^3$ , and  $\overline{g^2}$  is the harmonic mean square of  $g$ , the surface tension correcting factor, averaged over the volume of bubbles per radius interval. For other frequencies  $K_a$  varies as  $\nu^2$ . Since  $g$  is greater than unity in this range, the attenuation produced by microscopic bubbles is very much less than that produced by the same volume of air in the form of resonant bubbles.

When many bubbles are present, specular reflection of sound may occur from a region in which the density of bubbles changes rapidly over distances small compared to the wave length, provided that on each side of the region the bubble density is uniform. The presence of bubbles changes the velocity of sound; the real and imaginary parts both combine to give a reflection coefficient  $r$ . When the bubble sizes are distributed about resonance, the change in the real part of the velocity is very small, but the imaginary part is appreciable. For a ray of sound in bubble-free water, incident normally on a plane surface, beyond which extends a region of uniform bubble density, the reflection coefficient is

$$r = \frac{\sqrt{1 + \sqrt{1 + (2.5 \times 10^4 u_r)^2}} - \sqrt{2}}{\sqrt{1 + \sqrt{1 + (2.5 \times 10^4 u_r)^2}} + \sqrt{2}} ,$$

where  $u_r$ , defined in the same way as before, gives the density of bubbles in the region from which the sound is reflected. It is

evident that when  $u_p$  changes from less than  $10^{-5}$  to more than  $10^{-4}$  at a sharp discontinuity, most of the incident sound will be reflected even though for each bubble the scattering cross-section  $\sigma_s$  is much less than the absorption cross-section  $\sigma_a$ .

Bubbles of sizes far from resonance may also contribute to the reflection of sound in some situations. The relevant formulae are discussed in Sections IV and V.

It should be noted that all numerical statements and formulae in this Summary refer to air bubbles in water; also, except in Table I, a hydrostatic pressure of one atmosphere has been assumed in all cases. In other situations the more general equations derived in the following paper must be used.



Office of Scientific Research and Development  
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ACOUSTIC PROPERTIES OF  
GAS BUBBLES IN A LIQUID

by

Lyman Spitzer, Jr.

Columbia University  
Division of War Research  
At 172 Filson Street  
New York, N. Y.

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## INTRODUCTION

Air bubbles in water scatter and absorb underwater sound to a much greater extent than their geometrical radii might indicate. This effect arises from the fact that a small bubble resonates to sound whose wave length is several hundred times the diameter of the bubble. The scattering cross-section at resonance is several hundred times as great as the actual geometrical cross-section of the bubble. Since, in addition, a resonant bubble has a large amplitude of oscillation and dissipates a considerable amount of energy, the absorption cross-section of a bubble will be in many cases comparable to or greater than the scattering cross-section.

It is for this reason that bubbles have a considerable importance in the transmission of underwater sound. Even a very few bubbles, so widely scattered as to be almost invisible, may have an appreciable acoustic effect, and if the number of small bubbles is high, so that their presence is readily visible, the water will in many cases be very nearly opaque acoustically.

It may be useful to have assembled in one report all the present available detailed information on the acoustic properties of such bubbles. Calculations of the scattering and absorption to be expected have been carried out by several authors, including German, Japanese, English, and American scientists. In addition, observational data on this subject make possible an evaluation of the theoretical predictions. While there is still

some uncertainty concerning the exact values of the scattering and absorption to be expected in certain cases, and while additional observational evidence would be highly desirable, on the whole the theory summarized in this report should provide a reasonably accurate guide for the effects to be expected in the most important cases.

In the first section of the report, a detailed analysis is given of the scattering to be expected from a small bubble of gas in which there is no conduction of heat, surrounded by a fluid in which the viscosity is zero, and at the surface of which there is no surface tension. This analysis, which is fairly simple, is given in detail to illustrate the principles involved. In the second section this analysis is extended to include the effects of viscosity in the water as well as the effects of heat conduction and surface tension. The experimental data on the oscillations of a bubble in the sound field are presented in Section III.

The scattering, absorption, and reflection to be expected from many bubbles of the same size are analyzed in the fourth section of this report. In the following section, the effects produced by many bubbles of different sizes are also considered, and the absorption, scattering, and reflection shown to arise primarily from bubbles near resonance. The results of this last section will be the ones applicable in most practical situations.

I. SCATTERING OF SOUND BY A SINGLE IDEALIZED BUBBLE.

The scattering of sound from a small bubble in a liquid can be treated in one of several ways. The most general and the most elegant treatment is that which has been used in atomic and nuclear collision theory, and which Epstein<sup>1</sup> has employed in his discussion of the effect of viscosity on the scattering and absorption by small spheres. In this method, the velocity potential for the incident wave is expanded in a series of terms, each of which represents a standing wave in spherical coordinates, whose origin is at the center of the bubble. If the equation for the oscillation of the bubble is solved in spherical coordinates, the solution turns out to be another series of such terms. From a comparison of these two series and a consideration of the appropriate boundary conditions, both the scattered and the absorbed radiation may be determined. While this method has the advantage of generality, its meaning is not always physically clear, and the necessary analysis is more complicated than is required for the present purpose.

Since the radius of the bubble in all relevant cases is considerably less than the wave length of the incident sound, another method of analysis, which has been used by Willis<sup>2</sup>, is possible. In this method of analysis, the bubble is assumed to -----

1. P. S. Epstein, Th. Von Karman Anniversary Volume, 1941, p. 162.

2. Willis, British Report, reprinted as Confidential Report Section C4-BrTs-503, Dissipation of Energy Due to Presence of Air Bubbles in the Sea.

be in a uniform, but alternating pressure field, and the velocity associated with the incident wave is neglected. This approximation is equivalent to replacing the incident wave by the first term in the expansion discussed above. This treatment is valid provided that the radius of the bubble is very small compared to the wave length  $\lambda$ .

The physical picture associated with this analysis is that the bubble cannot be in equilibrium with an oscillating pressure unless the bubble itself is pulsating. The magnitude of this oscillation, and the amount of radiation scattered, is determined by the boundary condition that the pressure and velocity just outside the bubble must be the same as those just inside. To express these conditions in a more quantitative form, expressions must be introduced for the pressure and velocity both inside and outside the bubble. Let  $p_0$  represent the pressure in the incident sound wave, which in this approximation is taken to be a function of the time only, not of position. The dependence of  $p_0$  on time is given by the equation

$$p_0 = p_0' e^{i\omega t}, \quad (1-1)$$

where  $\omega$  is the angular frequency of the sound wave. In general, primed quantities will be used throughout to denote the value of a particular quantity when  $t$  is equal to zero.

The pressure inside the bubble may be denoted by

$P_0 + p_1$ , where  $P_0$  is the hydrostatic pressure. In general, constant pressures will be denoted by capital letters, while small letters will be used for oscillating pressures. To the same degree of approximation as before  $p_1$  may be assumed to be the same at all points inside the bubble at any one time, provided that the wave length of the sound in air is much greater than the radius of the bubble. The oscillation of the bubble will produce an external velocity  $v_e$ , and an external pressure  $p_e$  which must be added to  $P_0 + p_0$  to give the total pressure in the liquid. Both  $v_e$  and  $p_e$  will vary with  $r$ , the distance from the center of the bubble.

In this notation the two boundary conditions, expressing the equality of pressure and velocity on the two sides of the bubble surface, become

$$p_0 + p_e(R) = p_1 \quad , \quad (1-2)$$

$$v_e(R) = \frac{dR}{dt} \quad , \quad (1-3)$$

where  $R$  is the radius of the bubble. In addition,  $v_e$  and  $p_e$  are determined from a velocity potential  $\phi$  by the usual equations

$$v_e = \frac{\partial \phi}{\partial r} \quad , \quad (1-4)$$

$$p_e = -\rho \frac{\partial \phi}{\partial t} \quad , \quad (1-5)$$

For spherically symmetrical oscillations we have

$$\phi = \frac{A}{r} e^{i(\omega t - kr)} \quad , \quad (1-6)$$

yielding for  $v_e$  and  $p_e$  the expressions

$$v_e = \frac{-A(1+ikr)}{r^2} e^{i(\omega t - kr)}, \quad (1-7)$$

$$p_e = \frac{-i\omega A e}{r} e^{i(\omega t - kr)}, \quad (1-8)$$

where  $k$  is  $2\pi/\lambda$  and  $A$  is a constant;  $r$  is of course the distance from the center of the bubble.

Since  $dR/dt$  may be expressed in terms of  $p_1$  if the nature of the gas in the bubble is known, the boundary conditions (1-2) and (1-3) may be used to eliminate  $p_1$  and to determine the constant  $A$  in equation (1-6). It is clear that this constant gives the intensity of the scattered radiation. Since the flow of energy  $H$  per square centimeter corresponding to a sound pressure  $p$  is

$$H = \frac{\overline{p^2}}{\rho c},$$

where  $c$  is the velocity of sound and where the bar denotes an average over time, it follows from equation (1-8) that the total flow of energy in the outgoing or scattered wave at a distance  $r$  is

$$4\pi r^2 H_s = \frac{4\pi}{\rho c} \times \frac{\omega^2 |A|^2}{2}, \quad (1-9)$$

where  $|A|$  denotes the absolute value of the complex quantity  $A$ , and  $H_s$  is the flow of energy per unit area in the scattered wave. We shall let  $\sigma_s$  denote the "cross-section" of the bubble for scattered radiation. In physical terms, the energy appearing in



the scattered radiation is equal to the amount of energy in the incident beam which passes through an area  $\sigma_s$  perpendicular to the beam. In the present case, the energy in the incident beam passing through unit area is  $p_0'^2/2\rho c$ , and therefore

$$\frac{\sigma_s p_0'^2}{2\rho c} = 4\pi^2 H_s, \quad (1-10)$$

and with the use of equation (1-9), this gives

$$\sigma_s = \frac{4\pi^2 \omega^2}{p_0'^2} |A|^2. \quad (1-11)$$

We may also introduce a coefficient of extinction  $\sigma_e$ , defined as the cross-section for extinction of the incident radiation. The extinction includes both scattering and absorption and may be determined from the total work done on the bubble by the incident sound wave. The work done by  $p_0$  on the bubble per unit time, per unit area may be expressed as the product of the real parts of  $p_0$  and of the velocity  $dR/dt$  of the bubble surface. Since  $p_0'$  is real by definition, the average rate of work done on the bubble is

$$W = -4\pi R^2 p_0' \overline{\cos \omega t \mathcal{R}(dR/dt)}, \quad (1-12)$$

where  $\mathcal{R}$  denotes the real part of the following quantity. From equation (1-3)  $dR/dt$  equals  $v_e(R)$ ; to the same approximation as before we may write in equation (1-7)

$$(1+ikR)e^{-ikR} = 1 + k^2 R^2. \quad (1-13)$$

Since  $k^2 R^2$  is of the same order as terms already neglected,  $dR/dt$  becomes

$$\frac{dR}{dt} = v_e(R) = \frac{-A e^{i\omega t}}{R^2}, \quad (1-14)$$

and the real part of  $v_e(R)$  then becomes  $-A \cos \omega t / R^2$ , plus a term in  $\sin \omega t$ . If this value is substituted in equation (1-12) we have

$$W = 4\pi p_0 A \overline{\cos^2(\omega t)}, \quad (1-15)$$

where the term in  $\sin \omega t$  has been neglected, since its average value is zero. The intensity of the incident sound multiplied by the "extinction" cross-section  $\sigma_e$ , is equal to the total work done on the bubble. Hence as in equation (1-10)

$$W = \frac{\sigma_e p_0'^2}{2\rho c}. \quad (1-16)$$

Since the average value of  $\cos^2 \omega t$  is  $1/2$ , we have finally for  $\sigma_e$ ,

$$\sigma_e = \frac{4\pi \rho c}{p_0'} A^2. \quad (1-17)$$

Equations (1-11) and (1-17) are quite general, and may be applied in any case such that  $2\pi R/\lambda$  is much less than unity.

It remains only to find the value of  $A$ . This may be done by the use of the two boundary conditions (1-2) and (1-3). First, however, we must express  $\frac{dR}{dt}$  in terms of  $p_1$ . This relationship follows from the assumed equation of state of the gas in the bubble. As before we may write

$$p_1 = p_1' e^{i\omega t} \quad (1-18)$$

If the conditions are assumed to be wholly adiabatic, that is if the conduction of heat from the bubble into the water is neglected, and if the amplitude of oscillation of the bubble is small, we have the equation

$$dp_1/P_0 = -\gamma dV/V, \quad (1-19)$$

where  $V$  is the volume of the gas in the bubble. If isothermal conditions are assumed,  $\gamma$  must be omitted from equation (1-19). The rate of change of  $V$  is obviously

$$dV/dt = 4\pi R^2 dR/dt, \quad (1-20)$$

where  $R$  is again the radius of the bubble. Equation (1-18) may be differentiated to yield

$$\frac{dp_1}{dt} = i\omega p_1' e^{i\omega t} \quad (1-21)$$

Equation (1-19) may be written

$$\frac{1}{P_0} \frac{dp_1}{dt} = \frac{-\gamma}{V} \frac{dV}{dt} \quad (1-22)$$

If equations (1-20) and (1-21) are substituted in equation (1-22), we have

$$\frac{dR}{dt} = \frac{-i\omega V p_1'}{4\pi\gamma R^2 P_0} e^{i\omega t} \quad (1-23)$$

Since the volume  $V$  is given by

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$$V = \frac{4\pi R^3}{3} \quad , \quad (1-24)$$

equation (1-23) may be written in the form

$$\frac{dR}{dt} = \frac{-i\omega R p_1'}{3\gamma P_0} e^{i\omega t} \quad . \quad (1-25)$$

We may now write down the boundary conditions (1-2) and (1-3) in terms of  $p_1'$  and  $A$ . Using equation (1-1) for  $p_0$ , equation (1-8) for  $p_e$ , equation (1-18) for  $p_1$ , equation (1-14) for  $v_e$  and equation (1-25) for  $dR/dt$ , we have

$$p_1' = p_0' - \frac{i\rho\omega A}{R} e^{-ikR} \quad (1-26)$$

$$\frac{-A}{R^2} = \frac{-i\omega R p_1'}{3\gamma P_0} \quad . \quad (1-27)$$

These two equations may be used to eliminate  $p_1'$  and to solve for  $A$ , which becomes

$$A = \frac{-iR p_0' \rho\omega}{e^{-ikR} - \frac{3\gamma P_0}{\rho\omega^2 R^2}} \quad (1-28)$$

If we define the following quantities:

$$\omega_0^2 = \frac{3\gamma P_0}{\rho R^2} \quad , \quad (1-29)$$

and

$$a = kR = \frac{2\pi R}{\lambda} \quad , \quad (1-30)$$

and if we expand the exponential in equation (1-28), retaining

only the first term, A becomes

$$A = \frac{iRp_o' / \rho\omega}{\frac{\omega_o^2}{\omega^2} - 1 + ia} \quad (1-31)$$

If equation (1-31) is used in expression (1-11) for  $\sigma_s$ , we find

$$\sigma_s = \frac{4\pi\omega^2\rho^2}{p_o'^2} \times \frac{R^2 p_o'^2}{\omega^2\rho^2} \times \frac{1}{\left(\frac{\omega_o^2}{\omega^2} - 1\right)^2 + a^2}, \quad (1-32)$$

which yields

$$\sigma_s = \frac{4\pi R^2}{\left(\frac{\omega_o^2}{\omega^2} - 1\right)^2 + a^2} \quad (1-33)$$

The value of  $\sigma_e$  found from equation (1-17) is identical with that found for  $\sigma_s$ , since in this ideal case there is no absorption.

Since k equals  $\omega/c$ , equations (1-29) and (1-30) may be combined to give the result

$$a = \frac{\omega}{\omega_o} a_o, \quad (1-34)$$

where

$$a_o = \left\{ \frac{3\gamma P_o}{\rho c^2} \right\}^{1/2} \quad (1-35)$$

The quantity  $a_o$  is the value of a, or  $2\pi R/\lambda$ , at resonance; for air bubbles <sup>sea</sup>in water at atmospheric pressure at 60°F, equation (1-35) yields

$$a_o = 1.36 \times 10^{-2}, \quad (1-36)$$

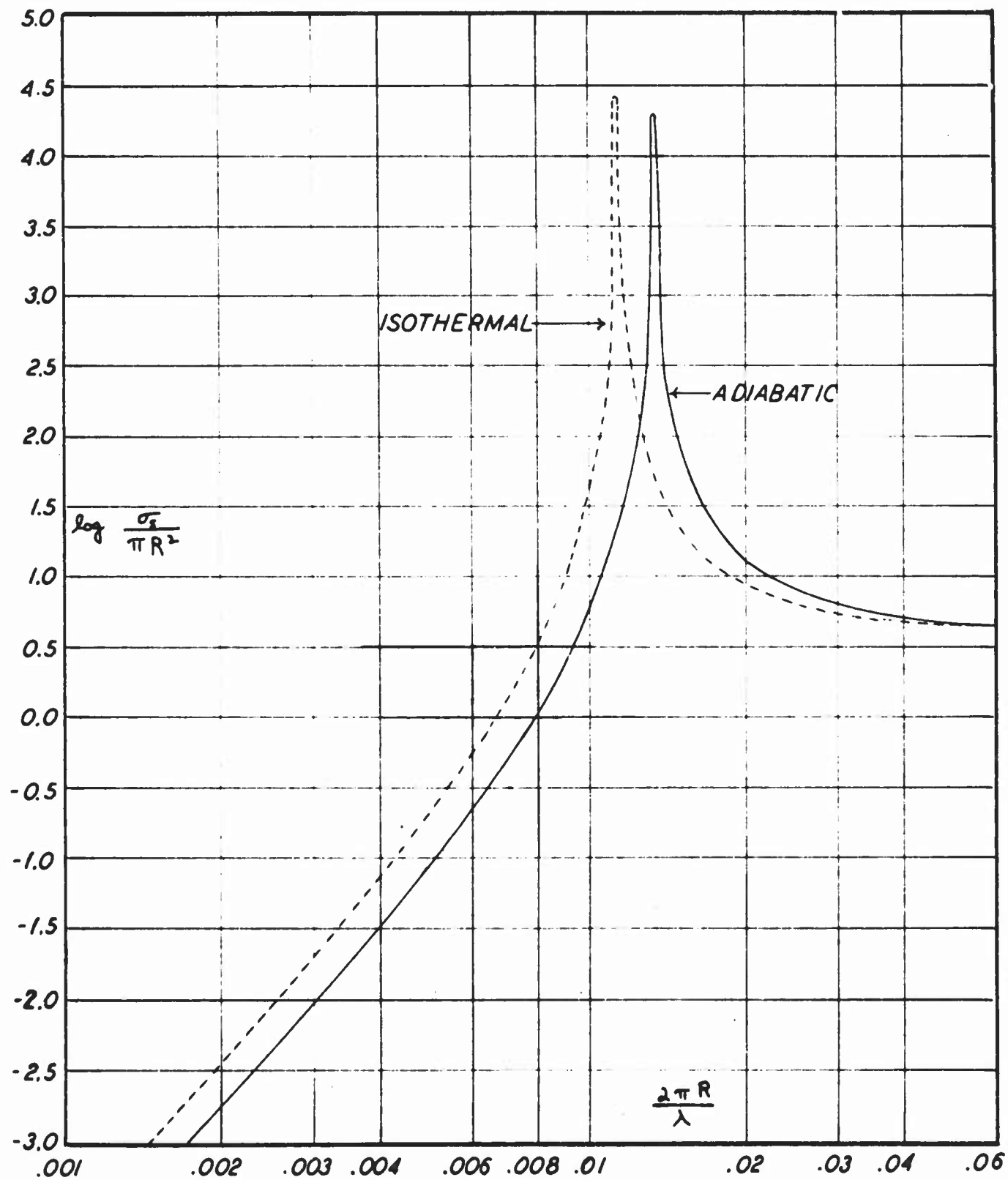
calculated for a sound velocity of  $1.49 \times 10^5$  cm/sec.

Equation (1-33) is represented graphically in Figure 1, where  $\log \sigma_s / \pi R^2$  is plotted against  $a$ . The solid line represents the adiabatic case, with  $\gamma = 1.4$ ; the dashed line represents the isothermal case, in which  $\gamma$  is replaced by unity. From equation (1-29) it is evident that the resonant frequency is decreased by a factor of  $\gamma^{1/2}$  when isothermal conditions replace the adiabatic ones. Since, moreover,  $\sigma_s$  is equal to  $4\pi R^2 (\omega/\omega_0)^4$  in the long-wave-length, or Rayleigh, region - see equation (1-33) - the scattering cross-section is greater in the isothermal case by a factor of  $\gamma^2$ . The details of the transition between adiabatic and isothermal conditions are contained in the results of the next section. Figure 1 is essentially the same as one given by Duvall.<sup>3</sup>

It is evident from Figure 1 that  $\sigma_s$  is enormously greater at resonance than it is elsewhere. From equations (1-30) and (1-33) it may be seen that the value of  $\sigma_s$  at the resonant peak is  $\lambda^2/\pi$ , corresponding to scattering of the energy incident on a sphere of radius  $\lambda/\pi$ . Since  $a$  is  $2\pi R/\lambda$ , it is evident from equation (1-36) that  $\lambda$  is roughly 400 times the value of  $R$  at resonance; the scattering of sound from an ideal bubble is in some circumstances enormously greater than would be expected from the geometrical cross-section of the bubble. It should be noted that

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3. Memorandum For File - M40 - University of California, Division of War Research, Scattering of Underwater Sound by Solid Particles and Air Bubbles. by George E. Duvall, February 11, 1943.

FIGURE 1  
SCATTERING CROSS-SECTIONS  
FOR IDEAL BUBBLES



the radius at which a bubble resonates may also be deduced<sup>3</sup> from relatively simple considerations, involving only the effective mass of the water immediately surrounding the bubble and the stiffness of the air in the bubble.



## II. SCATTERING AND ABSORPTION OF SOUND BY AN ACTUAL BUBBLE.

The results of the previous section are valid only in the ideal case, in which there is no dissipation of energy. In an actual bubble oscillations will be accompanied by a loss of energy, partly through viscosity, and partly through the loss of heat from the bubble into the fluid. In addition, the presence of surface tension will modify the results. The boundary conditions in the preceding sections may readily be modified to take these effects into account.

The presence of heat conduction modifies the equation of state so that  $dp_1/P_0$  is no longer equal to  $-\gamma dV/V$ . As pointed out in the last section, when the conduction of heat is so complete that isothermal conditions prevail,  $dp_1/P_0$  is equal to  $-dV/V$ . In the intermediate case, however,  $dp_1/P_0$  is no longer in phase with  $-dV/V$ ; it is this difference in phase that gives rise to the dissipation of energy.

This effect may be simply described in physical terms. As the bubble is compressed the temperature rises; when the rise of temperature is appreciable, heat conduction is important and the bubble tends to cool off even before expansion has started. When maximum compression is reached, the temperature will be decreasing as heat flows from the bubble into the water. It is clear that the maximum temperature will be reached somewhat before maximum compression, and that the temperature of the bubble

will tend to be somewhat greater during the compression than during the expansion. Since for a given volume the pressure varies directly as the temperature, the pressure exerted on the bubble during the compression will be greater for a given radius than the corresponding pressure during the expansion. Hence the work done by the compression of the bubble will be less than the work done by the bubble when it expands. The difference represents a dissipation of energy, which appears as a net flow of heat into the water.

The analysis for this case has been carried through by Willis<sup>2</sup>. The results show that

$$\frac{dV}{V} = \frac{-(\alpha - i\beta)}{\gamma} \frac{dp_1}{P_1} \quad (2-1)$$

where  $P_1$  is the average pressure of the gas in the bubble; also

$$\alpha = 1 + \frac{3(\gamma-1)}{\kappa} \left( \frac{\sinh \kappa - \sin \kappa}{\cosh \kappa - \cos \kappa} \right) \quad (2-2)$$

$$\beta = \frac{3(\gamma-1)}{\kappa^2} \left\{ \kappa \frac{\sinh \kappa + \sin \kappa}{\cosh \kappa - \cos \kappa} - 2 \right\} \quad (2-3)$$

where

$$\kappa^2 = \frac{2 \rho_1 c_p R^2}{K} \quad (2-4)$$

In equation 2-4,  $\rho_1$  is the density of the gas within the bubble, while  $c_p$  and  $K$  are the specific heat per unit weight at constant pressure and the heat conductivity for the gas. Since  $c_p$  and  $K$  are independent of density,  $\kappa^2$  varies directly as  $\rho_1$ , or as  $P_1$ .

if  $\omega$  and  $R$  are constant. If  $\omega$  is constant, while  $R$  satisfies equation (1-29) for a resonant bubble,  $K^2$  varies as  $P_0^2$ . Results similar to those of Willis<sup>2</sup> were found independently by Pfriem<sup>4</sup> and Saneyosi<sup>5</sup>. Certain of the assumptions made by Willis and others have been examined critically by Herring<sup>6</sup>, who finds that they are valid in the cases of practical importance.

When  $K$  is equal to or less than 2,  $\alpha$  and  $\beta$  are given to within one percent by the expansions

$$\alpha = \gamma - \frac{(\gamma-1)K^4}{630}, \quad (2-5)$$

$$\beta = \frac{(\gamma-1)K^2}{30} \left\{ 1 - \frac{K^4}{420} \right\}, \quad (2-6)$$

while for  $K$  equal to or greater than 5 we have, to the same accuracy,

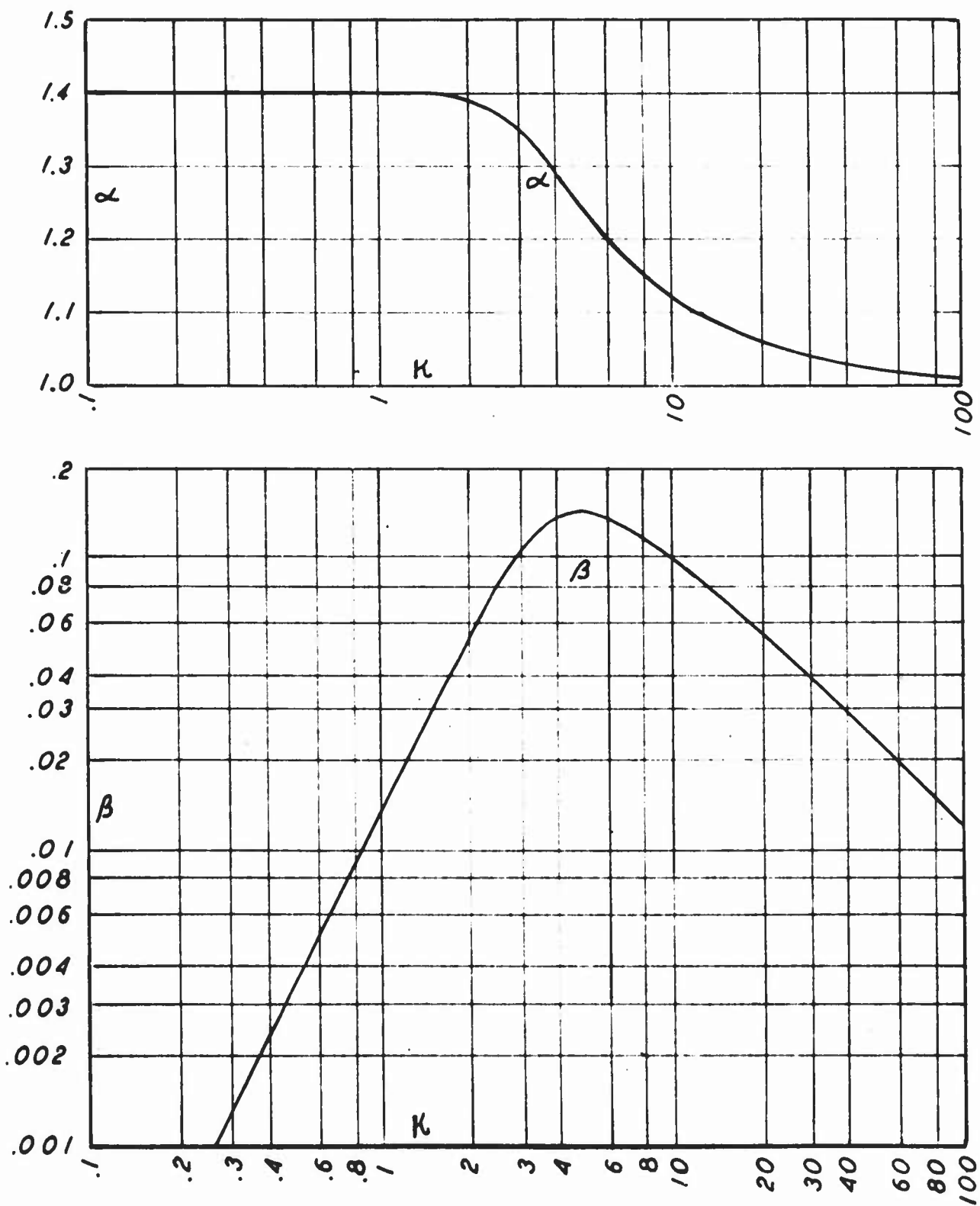
$$\alpha = 1 + \frac{3(\gamma-1)}{K}, \quad (2-7)$$

$$\beta = \frac{3(\gamma-1)}{K} \left\{ 1 - \frac{2}{K} \right\}. \quad (2-8)$$

Values of  $\alpha$  and  $\beta$  for a wide range of  $K$  are shown in Figure 2, taken from Herring's paper<sup>6</sup>. As expected,  $\alpha$  increases from 1 to  $\gamma$  as  $R$  decreases, while  $\beta$ , which gives the dissipation arising from heat conduction, has its maximum value in the transition region and vanishes for both very large (adiabatic) and very small (isothermal) bubbles.

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4. H. Pfriem, Akustische Zeitschrift, 5, 202, 1940.
  5. Z. Saneyosi, Electrotechnical Journal, 5, p. 49, 1941.
  6. C. Herring, Comments on the Report "Dissipation of Energy due to Presence of Air Bubbles in the Sea," by Willis.

FIGURE 2  
VALUES OF  $\alpha$  AND  $\beta$



The presence of viscosity in the surrounding fluid also produces a difference of phase between the pressure and the velocity at the surface of the bubble, with the result that more energy is required to compress the bubble than is regained in the subsequent expansion.<sup>7</sup> In the presence of viscosity, momentum is transmitted from one region of a fluid to another moving at a different velocity; an element of fluid moving rapidly in a particular direction tends to transmit its momentum to other elements of the fluid. In the present case, the effect of viscosity is perhaps difficult to visualize, since the viscous forces are approximately zero both inside and outside the bubble. The viscous stresses, which give the flow of momentum in different directions, do not vanish in the liquid outside the bubble, however. The point is that although momentum is flowing through the liquid, each small element receives as much as it loses, so that there is no net force on any small element of the liquid.

At the surface of the bubble, however, the presence of viscosity in the liquid will produce a flow of momentum across the surface into the gas, and this flow will not in general be equal to the corresponding flow in the reverse direction. This flow must be counterbalanced by an equal but opposite difference between  $p_1$  and  $p_0 + p_e$ , from which the dissipation of energy may

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7. An evaluation of the viscous damping of air bubbles in water was given by A. Mallock, Proc. Roy. Soc. A, 84, p. 391, 1910. His formula for the dissipation of energy is apparently in error and should be multiplied by a factor of  $4\pi/3$ .

be calculated.

More quantitatively, in cartesian coordinates the stress tensor  $p_{st}$ , which is the flow in the  $s$  direction of momentum in the  $t$  direction may be written<sup>8</sup>

$$p_{st} = - \left( p + \frac{2}{3} \mu \sum_{\lambda=1}^3 \frac{\partial v_{\lambda}}{\partial x_{\lambda}} \right) \delta_{st} + \mu \left( \frac{\partial v_s}{\partial x_t} + \frac{\partial v_t}{\partial x_s} \right), \quad (2-9)$$

where  $p$  is the total pressure and where  $s$  and  $t$  may each assume the values 1, 2, and 3, corresponding to the  $x$ ,  $y$ , and  $z$  axes, respectively; also

$$\delta_{st} = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases}, \quad (2-10)$$

In the present case the velocity is wholly radial and it is only the flow of radial momentum across the surface that is important; this quantity may be denoted by the symbol  $Q$ . Equation (2-9) then becomes

$$Q = -p - \frac{2\mu}{3r^2} \frac{\partial}{\partial r} (r^2 v) + 2\mu \frac{\partial v}{\partial r}. \quad (2-11)$$

Inside the bubble, the coefficient of viscosity  $\mu$  may be written  $\mu_1$ . If the temperature is assumed to be uniform, the bubble expands uniformly, and

$$v_1 = \frac{r}{R} \frac{dR}{dt}. \quad (2-12)$$

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8. Lamb, Hydrodynamics (Cambridge U. Press, 1932), p. 574.

With this substitution we find that  $Q$  is equal to  $-p$  inside the bubble, corresponding to the usual assumption that viscous effects vanish for uniform expansion or contraction of a fluid. If  $P_i$  denotes the average pressure inside the bubble, then

$$Q_i = -P_i - p_i \quad (2-13)$$

Outside the bubble, we may take equation (1-14) for  $v_e$  and since the total pressure  $p$  outside the bubble is the sum of the hydrostatic pressure  $P_o$ , the incident sound pressure  $p_o$ , and the scattered sound pressure  $p_e$ ,  $Q_e$  becomes

$$Q_e = -P_o - p_o - p_e + \frac{4\mu_e A e^{i\omega t}}{r^3} \quad (2-14)$$

At the surface of the bubble  $Q_i$  and  $Q_e$  would be equal except for the presence of surface tension, which contributes a term  $-2T/R$  to be added to  $Q_e$ , where  $T$  is the surface tension per cm at the liquid-gas interface. The boundary condition (1-2) now becomes

$$P_i + p_i = P_o + p_o + p_e(R) + \frac{2T}{R} - \frac{4\mu_e A e^{i\omega t}}{R^3}, \quad (2-15)$$

where the subscript  $e$  has been dropped from the coefficient of viscosity for the liquid.

When equation (2-15) is averaged over time, there results the familiar equation

$$P_i = P_o + 2T/R, \quad (2-16)$$

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expressing the fact that the surface tension increases the average pressure within the bubble. The time-dependent quantities in equation (2-15) yield the relationship

$$p_1' = p_0' + p_e'(R) + \frac{2Tp_1'}{3Rp_1} \frac{(\alpha - i\beta)}{\gamma} - \frac{4\mu A}{R^3}, \quad (2-17)$$

where equation (2-1) has been used to determine  $dR/dt$  in terms of  $p_1'$ , as in equations (1-19) through (1-25), and thus to determine the time-dependent part of  $1/R$ .

In equation (2-17) we may now use equation (1-8) to eliminate  $p_e'(R)$ , and equations (1-3), (1-14), (1-18), (1-20), and (2-1) to eliminate  $p_1'$ , and solve for  $A$ . Since  $\beta^2$  is always less than 2 percent of  $\alpha^2$ , we may neglect  $(\beta/\alpha)^2$  as compared to unity,<sup>9</sup> in which case  $A$  may be written in the form

$$A = \frac{iRp_0' / \rho\omega}{\frac{\omega_p^2}{\omega^2} - 1 + i\delta} \quad (2-18)$$

The resonant angular frequency  $\omega_r$  is given by

$$\omega_r^2 = \frac{3P_0\gamma}{R^2\rho\alpha} \left\{ 1 + \frac{2T}{Rp_0} \left( 1 - \frac{\alpha}{3\gamma} \right) \right\}, \quad (2-19)$$

or

$$\omega_r^2 = \omega_0^2 \frac{\beta}{\alpha}, \quad (2-20)$$

9. If  $(\beta/\alpha)^2$  is not neglected, the resonant peak occurs when  $(\omega/\omega_0)^2$  equals  $\alpha/(\alpha^2 + \beta^2)$  if surface tension effects are neglected. The contribution of heat conduction to the damping term  $\delta$  at resonance is accurately  $\beta/\alpha$ , however, as in equation (2-22). In the paper by Pfriem (ref. 4), Figures 2 and 3 give values of  $\beta/\alpha$  and  $\alpha/(\alpha^2 + \beta^2)$  as functions of  $K$ .



where  $\omega_0$  and  $\alpha$  are defined in equations (1-29) and (2-2), respectively, and where

$$g = 1 + \frac{2T}{RP_0} \left( 1 - \frac{\alpha}{3\gamma} \right) \quad (2-21)$$

When the surface tension  $T$  is negligible,  $g$  is equal to unity.

The "damping constant"  $\delta$  is given by

$$\delta = \frac{4\mu}{\rho\omega R^2} + \frac{\beta\omega b^2}{\alpha^2\omega^2} \left( 1 + \frac{2T}{RP_0} \right) + a \quad (2-22)$$

where  $a$  equals  $kR$ , or  $2\pi R/\lambda$ , as before. Since  $\delta$  is of primary importance near resonance, we shall be interested primarily in  $\delta_r$ , its value at resonance; from equations (2-20), (1-29), and (2-21), we find

$$\delta_r = \frac{4\mu\alpha\omega_r}{3\gamma P_0 g} + \frac{\beta}{\alpha h} + \frac{a g^{1/2}}{\alpha^{1/2}} \quad (2-23)$$

where

$$h = 1 - \frac{\alpha/3\gamma}{1 + RP_0/2T} \quad (2-24)$$

The quantity  $\delta_r$  is the relative half-width of the resonance peak obtained when either  $\sigma_s$  or  $\sigma_e$  is plotted against the frequency  $\nu$ ; i.e.,  $\pm \delta_r/2$  is the value of  $(\nu - \nu_r)/\nu_r$  at which  $\sigma_s$  and  $\sigma_e$  are each equal to one-half their peak values.

The scattering and extinction cross-sections may be found from equation (2-18) for  $A$  and from equations (1-11) and (1-17); we find

$$\frac{\sigma_s}{\pi R^2} = \frac{4}{\left( \frac{\omega_r}{\omega} - 1 \right)^2 + \delta^2} \quad (2-25)$$

$$\frac{\sigma_e}{\pi R^2} = \frac{4\delta/\omega}{\left(\frac{\omega_r^2}{\omega^2} - 1\right)^2 + \delta^2} \quad (2-26)$$

It is also convenient to define an absorption cross-section,  $\sigma_a$ , which is the difference between the extinction cross-section  $\sigma_e$ , and the scattering cross-section,  $\sigma_s$ . From equations (2-25), (2-26), and (2-22), it follows that

$$\frac{\sigma_a}{\pi R^2} = \frac{\frac{4}{a} \left\{ \frac{\beta \omega_r^2}{\alpha \omega^2} + \frac{4\mu}{\rho \omega R^2} \right\}}{\left(\frac{\omega_r^2}{\omega^2} - 1\right)^2 + \delta^2} \quad (2-27)$$

It may be shown that the first term in the numerator gives the same result as was found by Willis,<sup>2</sup> while the second gives that deduced by Epstein,<sup>10</sup> provided that surface tension is negligible so that both g and h may be set equal to unity.

These results may be illustrated by the specific case of an air bubble in water at a pressure of one atmosphere, for which the following constants may be used:

$$\begin{aligned} P_0 &= 10^6 \frac{\text{dynes}}{\text{cm}^2}, & \gamma &= 1.40, & \rho &= 1 \frac{\text{gm}}{\text{cm}^3}, \\ K &= 5.6 \times 10^{-5} \frac{\text{cal}}{\text{cm sec}}, & c_p &= 0.24 \frac{\text{cal}}{\text{gm}}, & \rho_1 &= 1.29 \times 10^{-3} \frac{\text{gm}}{\text{cm}^3}, \\ \mu &= 1.0 \times 10^{-2} \text{ poise}, & T &= 75 \frac{\text{dynes}}{\text{cm}}. \end{aligned}$$

10. P. S. Epstein, The Stability of Air Bubbles in the Sea, and the Effect of Bubbles and Particles on the Extinction of Sound and Light in Sea Water, NDRC Report No. C4-sr30-027, September 1, 1941.

With these values,  $k$ ,  $g$ , and  $h$  become

$$k = \frac{\omega}{\omega_0} \left( \frac{7.3 \times 10^3}{\nu(\text{kc})} \right)^2 ;$$

$$g = 1 + \left( 1 - \frac{\alpha}{4.2} \right) B \frac{\omega_0}{\omega} ;$$

$$h = 1 - \frac{\alpha/4.2}{1 + \omega/B\omega_0} ;$$

where

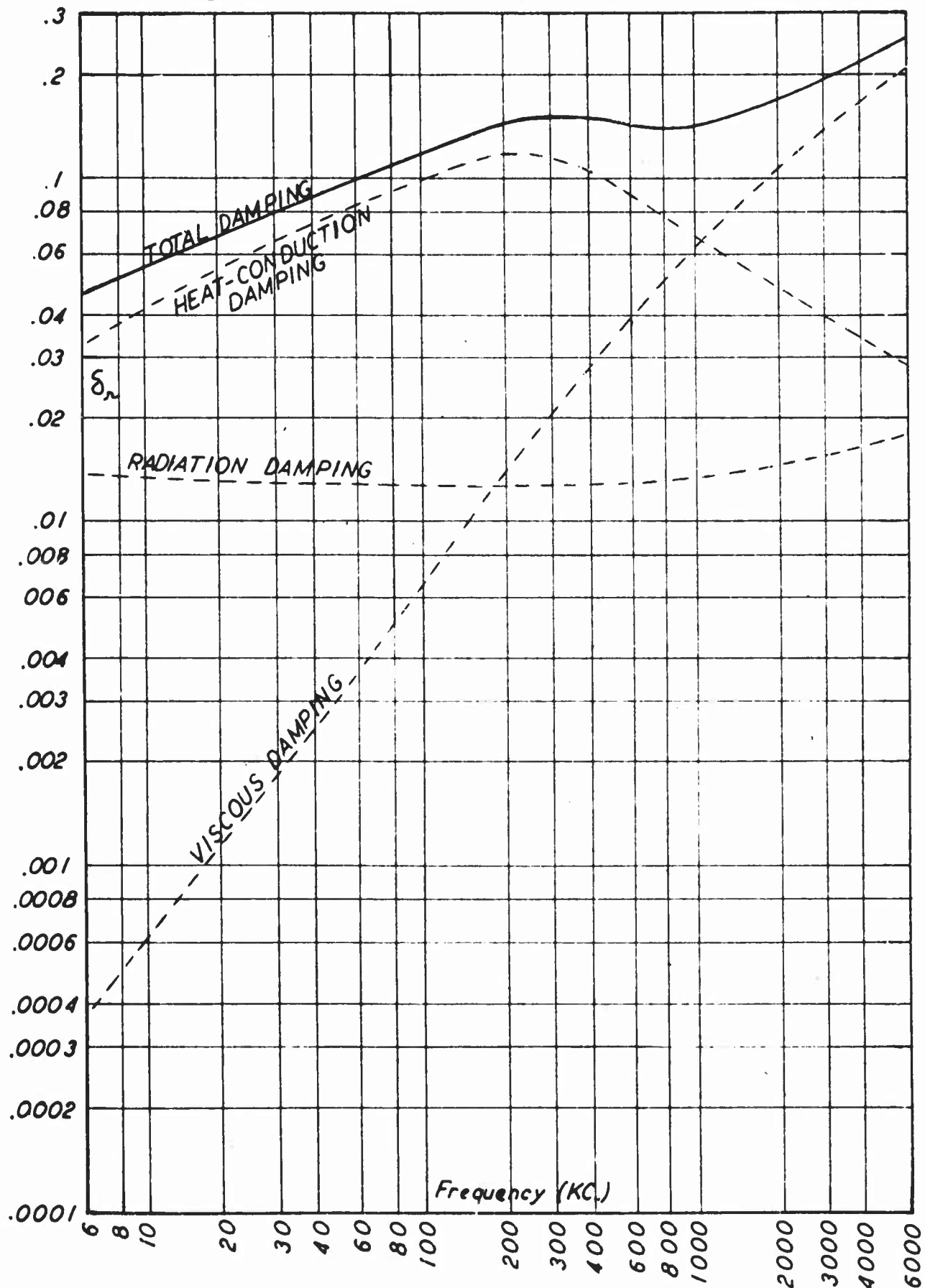
$$B = 4.6 \times 10^{-4} \nu(\text{kc}) ,$$

and  $\nu(\text{kc})$  is the frequency in kilocycles. The contribution of viscosity to  $\delta_r$  - equation (2-23) - becomes

$$\frac{4\mu\alpha\omega_r}{3\gamma P_0 g} = 6.0 \times 10^{-5} \frac{\alpha}{g} \nu_r(\text{kc})$$

The computations based on these values have been used to plot Figures 3-5. In Figure 3 are shown values  $\delta_r^{\text{st}}$  for different resonant frequencies, for an air bubble in water at atmospheric pressure. Since the extinction is proportional at resonance to  $\delta_r$ , it is useful to split  $\delta_r$  up into three parts, corresponding to the three terms in equation (2-23), each giving the extinction arising from a particular source. The three dashed curves represent the different terms in equation (2-23), while their sum, shown by the solid curve, represents the total value of  $\delta_r$ . When the hydrostatic pressure  $P_0$  is increased, the radiation damping increases as  $P_0^{1/2}$ , the viscous damping is inversely proportional to  $P_0$ , for a fixed resonant frequency, while the heat-conduction damping at resonance reaches the same maximum value but at a frequency which

FIGURE 3  
THEORETICAL DAMPING CONSTANTS  
FOR RESONANT BUBBLES



varies as  $P_0^2$ . For a bubble at 300 feet, corresponding to a pressure of 10 atmospheres, the damping constant  $\delta_r$  will be practically equal to .043, the value arising from radiation damping, for frequencies less than 100 kc.

It is evident that the effect of viscosity is quite negligible for resonant bubbles at frequencies much less than 1000 kc. At higher pressures the effect of viscosity is even smaller.

Figure 4 shows values of  $\log \tau_s / \pi R^2$  as a function of  $\omega / \omega_0$  for sound frequencies of 6000, 24,000, 200,000 and 5,000,000 cycles per second, and for a hydrostatic pressure  $P_0$  of one atmosphere. The increasing width and decreasing height of the resonance peaks as the frequency is increased is a result of the increase of  $\delta_r$  with frequency; the shift of the resonance peak to lower values of  $\omega / \omega_0$  and the increase of  $\sigma_s$  for low values of  $\omega / \omega_0$  as the frequency is increased results from the change in  $\alpha$  depicted in Figure 2, and is a result of the transition from adiabatic to isothermal conditions. For the highest frequency, the effect of surface tension becomes important in increasing the stiffness of the bubble, and the resonant angular frequency  $\omega_r$  becomes considerably greater than  $\omega_0$ .

In Figure 5 are plotted values of the logarithm of the absorption cross-section  $\sigma_a$ , divided by the geometrical cross-section  $\pi R^2$ , again as a function of  $\omega / \omega_0$  and for a pressure of one atmosphere. Near resonance, the curves are again determined

FIGURE 4  
SCATTERING CROSS-SECTIONS

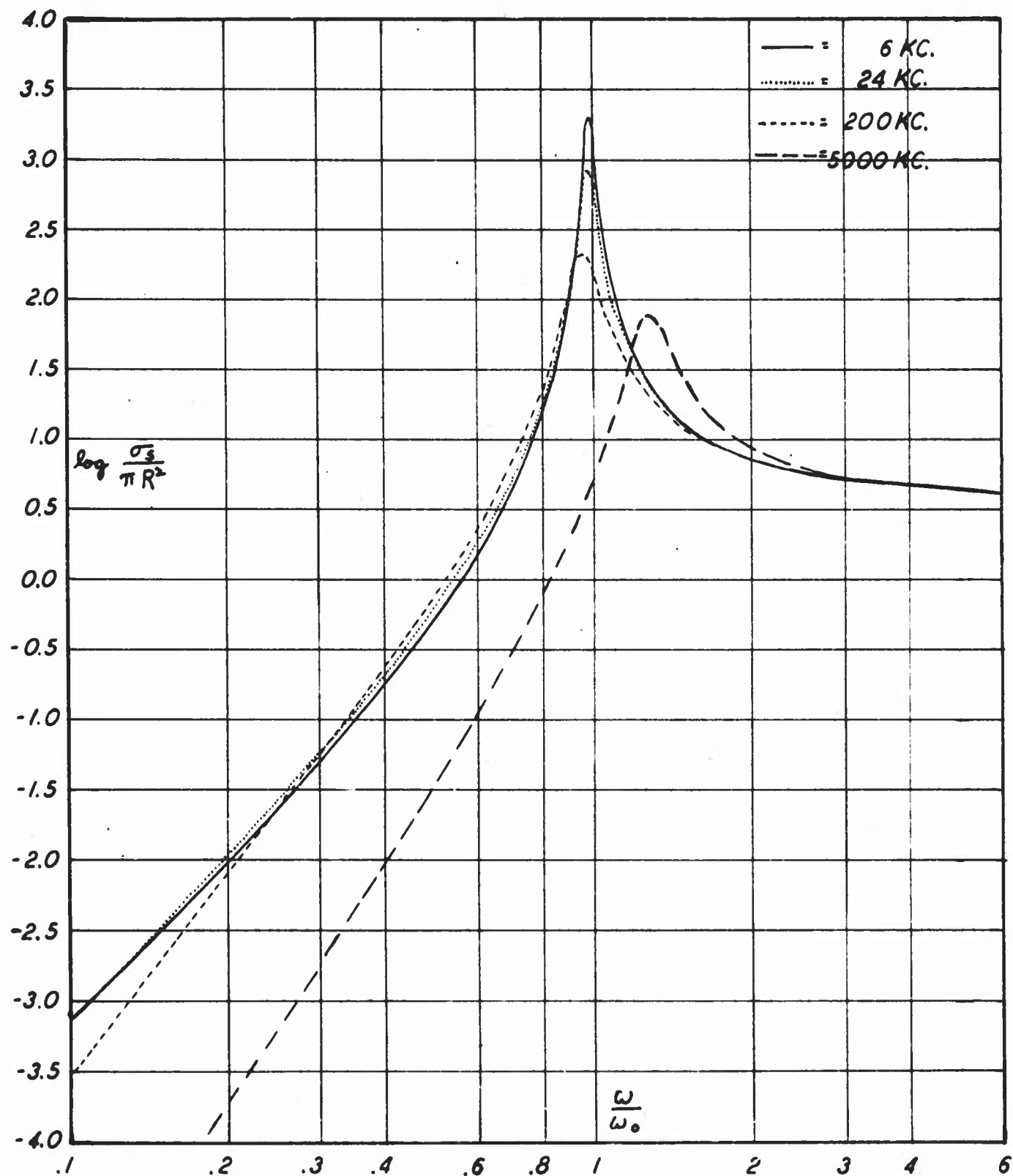
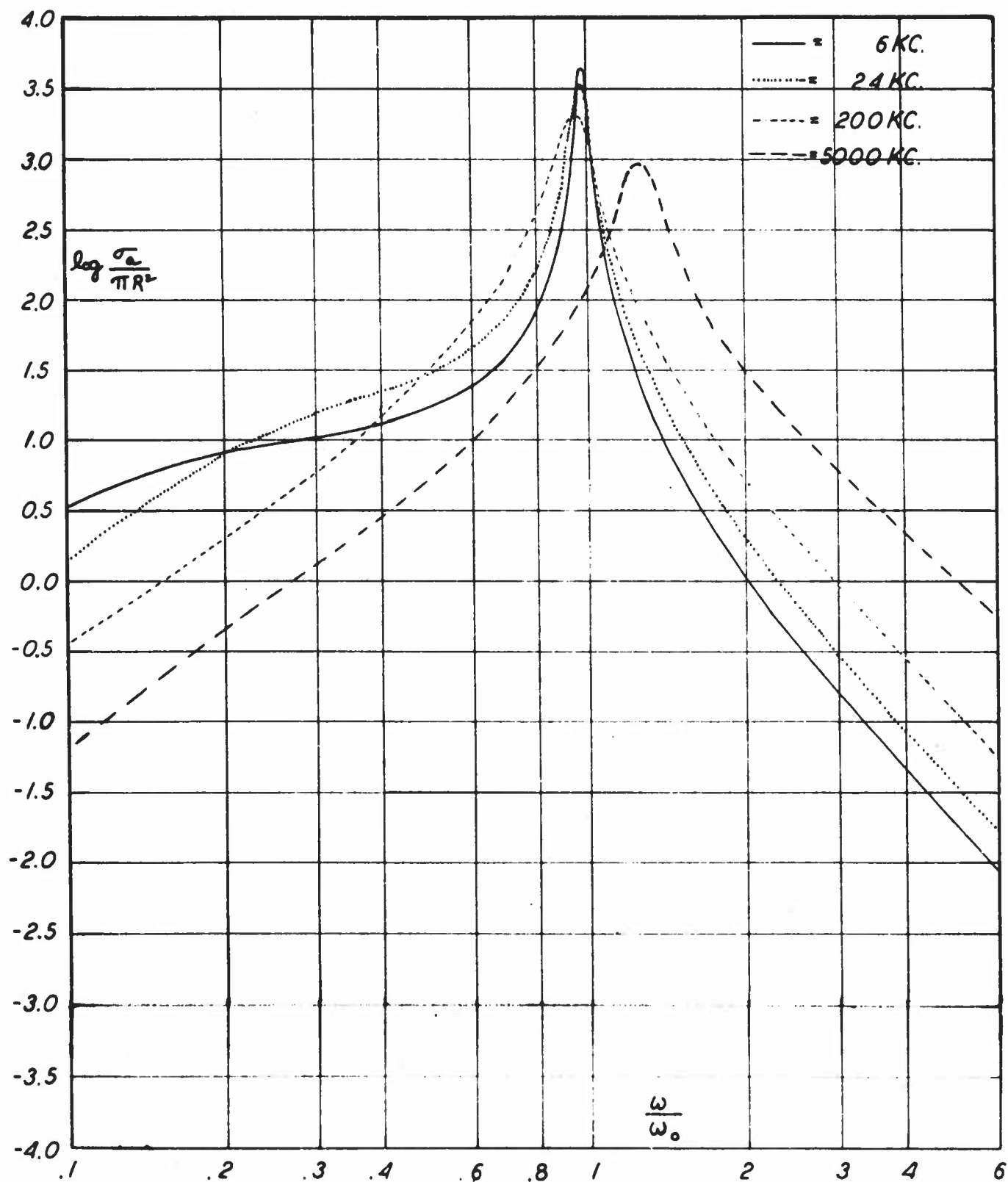


FIGURE 5  
ABSORPTION CROSS-SECTIONS



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by the values of  $\delta_r$ , shown in Figure 3; as  $\delta_r$  increases, the absorption cross-section at resonance actually decreases, but the resonance peak becomes correspondingly wider.

Far from resonance the value of  $\delta$  for other values of the frequency becomes important. The absorption is of course the sum of two effects, arising from heat conduction and viscosity, respectively, corresponding to the two terms in equation (2-27). For bubbles very much smaller than the resonant size, or for frequencies of several hundred kc or more,  $\kappa$  will be less than 2, equation (2-6) may be used for  $\beta$  and the ratio of these two terms becomes a function of the bubble radius  $R$  only, independent of the frequency. For an air bubble in water at a pressure of one atmosphere, the two terms are equal, and the absorption due to viscosity equals that due to heat conduction when  $R$  equals  $3 \times 10^{-4}$  cm, corresponding to a resonant frequency of  $10^3$  kc, or one megacycle. For bubbles greater than  $3 \times 10^{-4}$  cm in radius viscosity will be negligible compared to heat conduction, unless the bubble is so much larger than the resonant size that  $\beta$  again becomes very small. The absorption produced by such large bubbles is usually much less than the scattering (see Figures 4 and 5) and is not generally important.

The results derived in this Section are based on two fundamental assumptions, - firstly, that  $2\pi R/\lambda$ , the ratio of the bubble circumference to the wave length of sound is less than unity; secondly, that  $\Delta V/V$ , the relative change of volume of the



bubble, is small. The first condition restricts the analysis to relatively small bubbles. The chief interest lies in bubbles of resonant size or less, and since the radius of a bubble which resonates to sound of a particular frequency is much less than the wave length of the sound, this restriction is not serious. The second condition places a restriction on the intensity of the incident sound.

To examine this restriction quantitatively it is necessary to derive a value for  $\Delta V_{\max}/V$  where  $\Delta V_{\max}$  denotes the maximum value of  $\Delta V$  in the course of the pulsation. If  $p_i'$  is found from equation (2-17), substituting equations (2-18) for  $A$  and (1-8) for  $p_e'(R)$ , then equation (2-1) yields, after some simplification, and with the neglect of surface tension,

$$\frac{\Delta V_{\max}}{V} = \frac{p_o'}{P_o} \frac{\alpha}{\gamma} \frac{1}{\left\{ \left( 1 - \frac{\omega^2}{\omega_r^2} \right)^2 + \delta_r^2 \right\}^{1/2}} ; \quad (2-28)$$

in deriving this equation  $(\beta/\alpha)^2$  has been neglected compared to unity, and  $\delta^2 \omega^4 / \omega_r^4$  has been set equal to  $\delta_r^2$  in the denominator, since this term is important only near resonance.

From equation (2-28) it follows that when  $\omega$  is much less than  $\omega_r$  - that is, in the long-wave-length, or Rayleigh region, -  $\Delta V_{\max}/V$  is small if the sound pressure  $p_o'$  is much less than the hydrostatic pressure  $P_o$ . This is approximately the same condition that the sound-wave pressure be less than the cavitation limit. When  $\omega$  is much greater than  $\omega_r$ , the sound pressure  $p_o'$  can be as

great as  $P_0$  without making  $\Delta V_{\max}/V$  as great as unity.

At resonance, however,  $\Delta V_{\max}/V$  is equal to or greater than one, according to equation (2-28), when  $p_0'/P_0$  is roughly equal to or less than the damping constant  $\delta_r$ . From Figure 3 the value of  $\delta_r$  at 20 kc is about .08. As will be shown in the next Section, the observed value of  $\delta_r$  at this frequency is about .20. In either case, the relative change of volume is no longer small when the maximum incident sound pressure  $p_0'$  is one-tenth of the hydrostatic pressure  $P_0$ . When  $p_0'$  equals  $P_0$  the sound intensity is approximately 190 db above the reference level of .0002 dynes/cm<sup>2</sup>. Hence the theory developed here cannot be applied for resonant bubbles in a sound field above 170 db. At lower frequencies this limiting intensity becomes even less, owing to the decrease of  $\delta_r$  with decreasing frequency. Such intense sound fields are found only in the close neighborhood of sound projectors.

For resonant bubbles close to a powerful sound projector  $\Delta V_{\max}/V$  computed from equation (2-28) becomes large compared to unity and the phenomenon becomes quite different. In particular the oscillations are no longer purely harmonic, and begin to resemble those of a gas bubble produced by an explosion.<sup>11</sup> The scattered radiation will tend to be generated

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11. Theory of Pulsations of the Gas Bubble Produced by an Under-water Explosion, by C. Herring, NDRC Report No. C4-sr20-010, dated October 1941.

when the bubble is at its minimum diameter and will be composed of much higher frequencies than the incident sound wave, resembling a series of shock waves rather than a sinusoidal pressure wave. The theory of this phenomenon would presumably be quite different from that which has been developed in this paper.

### III. EXPERIMENTAL DATA ON THE OSCILLATION OF A SINGLE BUBBLE.

Some observations have been made on the oscillations of single bubbles in a sound field. The theory in Section II may be used to predict  $\nu_r$ , the frequency at which a bubble resonates, and also  $\delta_r$ , the relative half-width of the resonance peak found when the square of the amplitude of oscillation is plotted against frequency. Experimental values of these two quantities may, therefore, be used to check the validity of the theory.

The first examination of resonant bubbles was that of Minnaert,<sup>12</sup> who derived equation (1-29) and confirmed its accuracy for bubbles with diameters between 3 and 6 mm. The bubbles, when formed individually in a pail of water, produced musical tones with frequencies of 1000 to 2000 cycles per second. The pitch was determined by ear to within a fraction of half a tone, using as a comparison standard a tuning fork with a frequency of 264 cycles per second. The mean error of this comparison was estimated to be a fifth of a tone. The volume of each bubble was determined with a gas pipette.

Measurements with bubbles of different sizes in liquids of different densities confirmed accurately the theoretical relationship. As predicted, changes of temperature had no effect, while changes of the type of gas in the bubble affected the pitch only if  $\gamma$ , the ratio of the specific heats was changed. If the factor  $g/\alpha$  is set equal to unity in equation (2-20), and the

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12. M. Minnaert, Phil. Mag. 16, 235, 1933.

value of  $\omega_0$  is found from equation (2-29), with  $\gamma$  equal to 1.40, the resonant frequency  $\nu'_r$  in cycles per second may be expressed by the simple relationship

$$\nu'_r = \frac{328}{R} \left( \frac{P_0(\text{atm})}{\rho} \right)^{1/2}, \quad (3-1)$$

where  $R$  is the radius of the bubble in cm and the pressure  $P_0$  is expressed in atmospheres. For various gases with  $\gamma$  equal to 1.40, the weighted mean of all 64 observations gave a value of 330 for the constant in this equation. The agreement between theory and observation is excellent.

A later theoretical paper by Smith<sup>13</sup> discusses the effects of surface tension. Smith also points out that the amplitude of oscillation at resonance is so great that other sources of damping are probably important. The probable destructive effect of small resonant bubbles on any solid matter nearby is also briefly discussed.

Meyer and Tamm's<sup>14</sup> quantitative work at supersonic frequencies not only provides further confirmation of equation (3-1) for the resonant frequency, but also evaluates the damp-

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13. F. D. Smith, Phil. Mag. 19, 1148, 1935. Smith's equation takes account of the difference between  $P_i$  and  $P_o$ , the mean pressures inside and outside the bubble, but neglects the change in surface tension as the bubble oscillates, - see equation (2-17) - and is therefore not quite correct.

14. E. Meyer and K. Tamm, Akustische Zeitschrift, 4, 145, 1939. Available in English as Translation 109, David W. Taylor Model Basin, April 1943.

ing constant  $\delta_r$ . Two frequency ranges were investigated, from 1.5 to 6.5 kc, and from 21 to 30 kc.

In the lower frequency range, a photoelectric method was used for determining the amplitude of oscillation of the bubble. A single air bubble was illuminated optically and the scattered light measured by a photoelectric cell. When sound was produced by an electromagnetic telephone the oscillation of the bubble varied as the frequency varied and the alternating EMF produced by the photocell varied correspondingly. Variation of the sound pressure with frequency in the absence of the bubble was determined and allowed for by separate measurements with a hydrophone. The bubble radius was determined visually with the aid of a microscope. The results gave for each bubble the frequency of maximum amplitude as well as the half-width of the resonance peak.

For the higher frequency range, two methods were used, both employing an electrolytic method for producing the gas bubbles, and a microscope for measuring the radii. A visual optical method was employed to determine the frequency of maximum oscillation of a single bubble adhering to a small platinum electrode in water. This method could not be used to give a value for the relative half-width,  $\delta_r$ . For more accurate work, a single bubble was caught on a small wax sphere fastened to a platinum thread 1 cm long and  $1.5 \times 10^{-2}$  mm thick between the poles of an electromagnet. When the bubble oscillated in a sound field, the ribbon moved

back and forth and in the magnetic field; the alternating EMF generated was then amplified to give a measure of the amplitude of oscillation of the bubble. The presence of numerous resonance peaks at different frequencies, even in the absence of the bubble, made it difficult to attach much importance to the change of EMF with frequency. However, from the variation of amplitude with bubble radius at a given frequency, it was found possible to determine both the radius of the bubble at resonance and the width of the resonant peak.

The values of the resonant frequency for bubbles of air, hydrogen and oxygen in water were found to agree in all cases with equation (3-1) within the experimental error, which on the average was somewhat less than 5 percent. For a resonant bubble of oxygen at 30 kc  $\alpha$  equals 1.06, while  $g$  is 1.01; the resonant frequency, in accordance with equation (2-19), should be some 3 percent less than the value given by equation (3-1). This difference is too small to be shown in Meyer and Tamm's work. For a resonant hydrogen bubble at 24 kc, however,  $\alpha$  is 1.18, while  $g$  is still 1.01, and the resonant frequency should be 9 percent less than the value computed for a wholly adiabatic oscillation. The observed frequencies for hydrogen bubbles at frequencies of 27, 32, and 35 kc are only 3 to 4 percent less than the values given by equation (3-1). In view of the inaccuracy of the data, this discrepancy is not serious. One may conclude that observa-

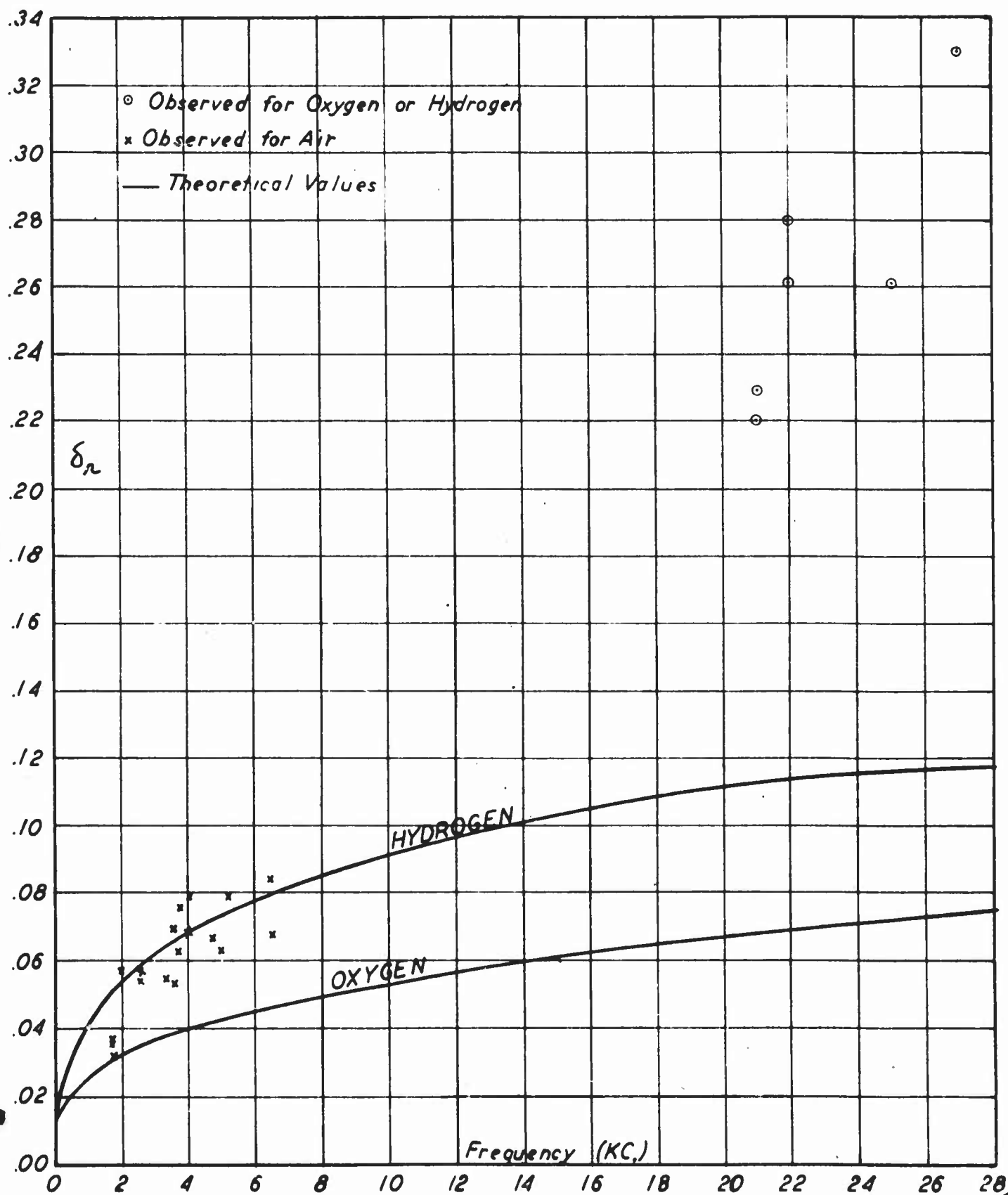
tions definitely confirm the approximate validity of equation (3-1) for frequencies from several thousand cycles up to some 20 kc or more, but that no adequate data are available to prove or disprove the more general equation (2-19).

The determinations of  $\delta_r$ , on the other hand, are in definite disagreement with the theory. As shown in Section II, the effect of viscosity should be wholly negligible for resonant gas bubbles in water at frequencies of 30 kc or less. Hence, when  $g$  is unity,  $\delta_r$  should be equal to the sum of  $a_0/\alpha^{1/2}$  and  $\beta/\alpha$ . Meyer and Tamm express their results in terms of  $\Theta_0$ , the logarithmic decrement per cycle of a freely resonating bubble. Since  $\Theta_0$  equals  $\pi\delta_r$ , the values of  $\delta_r$  are readily determined. The observed values of  $\delta_r$  and the theoretical curves for oxygen and hydrogen bubbles in water are shown in Figure 6. In the region 1.5 to 7 kc the values were found from air bubbles in water, which should in theory agree closely with the curve for  $O_2$  bubbles. At the higher frequencies, bubbles were produced electrolytically, and it is not stated by Meyer or Tamm whether the values of  $\delta_r$  refer to oxygen or hydrogen bubbles. The two theoretical curves differ because of the much greater heat conductivity for hydrogen than for oxygen.

The discrepancy between theory and observation is evident at once from this figure. The observed values of  $\delta_r$  are apparently much greater than can possibly be explained by heat conduction losses. The maximum value of  $\beta/\alpha$  is 0.115, which is the greatest contribution to  $\delta_r$  which heat-conduction losses can



FIGURE 6  
OBSERVED AND THEORETICAL  
DAMPING CONSTANTS



in theory provide, when surface tension effects are unimportant. Some other source of dissipation must probably be invoked to explain values of  $\delta_r$  as great as 0.25.

It is not certain whether this additional dissipation actually took place in and around the oscillating bubble, or whether it arose from the particular conditions of the experiments. The oscillation of the platinum thread in a magnetic field, for instance, may have produced considerable damping if the external resistance was not sufficiently large. On the other hand, it has been suggested by Pekeris<sup>15</sup> that a thin layer of high viscosity may exist at the surface of an air bubble in water. A high superficial viscosity is suggested by the fact that air bubbles rise through water at the same rate as solid particles of the same buoyancy. If such a layer exists, it might conceivably give rise to the observed high values of the damping constant  $\delta_r$ .

Figure 6 is very similar to Figure 3 in the paper by Saneyosi<sup>5</sup>, except that here the accurate values for  $\beta/\alpha$  have been used instead of the approximate formulae. A similar plot was published as Figure 4 by Pfriem<sup>4</sup>, but his theoretical curve is drawn too high, owing to an oversight in the derivation of his equation (21). Pfriem discusses other possible sources of dissipation and concludes that they are all negligible, except possibly for the periodic condensation and evaporation of water

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15. C. L. Pekeris, The Rate of Rise and Diffusion of Air Bubbles In Water, NDRC Report No. C4-sr20-326, October 22, 1942.

on "condensation centers" throughout the volume of the bubble.

The results of Meyer and Tamm on the value of  $\delta_r$  for air bubbles in glycerine could be used to check the theory if the viscosity of glycerine were accurately known. Unfortunately, this quantity varies markedly with the temperature, the value of which is not given. The observed values of  $\delta_r$  in this case vary from 0.083 at 2 kc to 0.118 at 4.6 kc. When the effects of heat conduction and radiation damping are allowed for, these values are consistent with a value of 4 for  $\mu$ , the coefficient of viscosity; for glycerine this corresponds to a temperature of about 80°F. The range in frequency is not sufficient to demonstrate whether or not the variation of  $\delta_r$ , when heat-conduction and radiation damping have been allowed for, is strictly proportional to the frequency as predicted by equation (2-23).

#### IV. MANY BUBBLES OF THE SAME SIZE.

When many bubbles are present in a sound field, two complexities arise. Firstly, interactions between bubbles may alter the amplitude of oscillation of each bubble. Secondly, the resultant sound field will be the sum of the sound waves emitted by all the bubbles, taking into account the phase of each wave. In certain simplified situations, however, an analytical solution is possible.

The simplest case arises when the distance between bubbles is always greater than the wave length of sound. In this situation a bubble will affect adjacent bubbles only in so far as it alters the value of the incident sound pressure  $p_0$  in their neighborhood, and thereby changes their amplitude of oscillation. In addition, there will be a large difference of phase between the waves from different bubbles, and on the average, the energy in the sound field may be found from the sum of the squares of amplitudes of the different waves. As a result, all questions of phase may be disregarded when the effects of different bubbles are combined, and all intensities may be added directly.

If there are  $n$  "widely spaced" bubbles per  $\text{cm}^3$  and if the scattering coefficient per bubble is  $\sigma_s$ , then the energy scattered by each bubble will be  $\sigma_s I$ , where  $I$  is the flux of energy in the incident sound beam, in  $\text{ergs per cm}^2$  per second.

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The total radiation scattered from a cubic centimeter will then be  $n\sigma_s I$ , provided  $n\sigma_s$  is small compared to unity; the quantity  $n\sigma_s$  may be regarded as the scattering cross-section per unit volume. If the effect of absorption is also taken into account, the intensity of energy in the main beam will then decline according to the law

$$I(x) = I(0)e^{-n(\sigma_s + \sigma_a)x} \quad , \quad (4-1)$$

provided that  $n(\sigma_s + \sigma_a)$  is constant. It is assumed in this equation that the beam is directed along the x axis.

Diffuse or multiply scattered radiation may, of course, replace the sound energy in the direct beam. In fact if no sound energy is absorbed the total flux of energy H must remain constant. For the flow of radiation through a scattering layer of thickness X a simple approximate solution<sup>16</sup> is available for  $I(x)$ , the average at the point x of the radiation intensity in all directions:

$$\overline{I(x)} = 2H(1 + \frac{3}{2}\tau) \quad , \quad (4-2)$$

where

$$\tau = \int_x^X n\sigma_s dx \quad . \quad (4-3)$$

The radiation flows in the direction of increasing x, or decreasing  $\overline{I(x)}$ . If  $n\sigma_s$  is constant, and  $\sigma_a$  is zero, the sound energy H

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16. A. S. Eddington, The Internal Constitution of Stars (Cambridge U. Press, 1926), p. 322. For a more accurate picture, the "equation of transfer" may be solved by successive approximations.

which leaves the far side of the layer is approximately

$$H = \frac{I(0)}{3n\sigma_s X} \quad (4-4)$$

If  $n\sigma_s X$  is appreciable, the amount of radiation penetrating through the bubble layer by multiple scattering will be very much greater than that computed from the exponential relationship in equation (4-1). Hence, absorption provides a much more effective sound screen than does scattering.

The effect of bubbles on a sound field may also be computed in certain cases when the average distance between bubbles is much less than a wave length. In this situation the sound waves emitted by adjacent bubbles will be in phase, and will interfere constructively. A small isolated group of bubbles crowded together in a sphere less than a wave length in radius will, for instance, scatter very much more sound than would be expected from a simple addition of intensities.

Before the effects of closely spaced bubbles can be combined, it is necessary to examine whether the interaction between such bubbles affects the validity of equations (2-25) and (2-27) for the scattering and absorption cross-sections. These cross-sections depend entirely on the amplitude of oscillation, which is proportional to the quantity  $A$ . The dependence of the amplitude of oscillation on the spacing between bubbles must be considered separately for non-resonant and for resonant bubbles.

For frequencies far from resonance, the formulae for

A developed in Section II should be valid even when the bubbles are fairly close together. At a distance of  $5R$  from such a bubble, the time-dependent part of the external pressure will be closely equal to  $p_0'$ , the pressure in the incident sound wave. The kinetic and potential energies of another oscillating bubble placed at this point should be very nearly equal to the corresponding energies for an isolated bubble with the same amplitude of oscillation. Hence, the oscillation of such a bubble should be largely unaffected by the presence of the other bubble  $5R$  away. In fact for wave lengths greater than resonance, the amplitude of oscillation is constant no matter how closely spaced the bubbles may be. For wave lengths shorter than resonance, however, the kinetic energy of oscillation and the external pressure producing the oscillation will both be affected if the bubbles are too close. The formulae in Sections I and II should be accurate to within 10 to 20 percent, however, when the average distance between adjacent bubbles is equal to  $5R$ . A mean separation of  $5R$  between closest neighbors corresponds to a bubble density of roughly 3 parts air in 100 parts water.

In the case of resonant bubbles, another effect must be considered. The fluctuating external pressure in the neighborhood of a resonating bubble is so different from  $p_0'$ , the pressure in the incident sound wave, that two neighboring bubbles may be expected to have a large mutual interaction. In fact when  $N$  bubbles are gathered together in a cluster small compared to the wave length, and no other bubbles are present, the radiative damping

constant for each is  $N$  times its value for an isolated bubble, corresponding to the fact that for a source much smaller than the wave length the emitted energy is proportional to the square of the radiating surface.

When a uniform distribution of bubbles is assumed, however, the contribution made by radiation damping to the value of  $\delta_r$  is unaffected by the close distance between two bubbles. The effects produced by the different bubbles give rise to a refracted wave whose velocity may differ from the sound wave in an undisturbed medium. If  $p_0$  is taken to be the pressure in the wave, resulting not only from the initial sound, but also from the superimposed wavelets emitted by the individual bubbles, then the difference in phase between  $p_0'$  and  $A$  will be correctly given by the analysis in Section II and the radiation damping will be the same as for an isolated bubble. This corresponds to the fact, discussed below, that the total scattered energy is unchanged by the spacing between the bubbles. Thus, even for resonant bubbles the amplitude of oscillation for a given sound pressure  $p_0$  will be unaffected by the presence of the other bubbles within distances greater than some  $5R$ , provided we include in  $p_0$  the wavelets emitted by all the other bubbles, and provided the bubbles are distributed with random uniformity.

Subject to all the limitations derived above, it is possible to compute the scattering by adding together the scattered waves produced by different bubbles. The resultant effect depends



on the density of bubbles as a function of position. The essential features of the problem will be treated if only two types of distribution are considered: first a random, uniform distribution; second a discontinuous change in the bubble density,

If the number of bubbles in each unit volume were exactly constant, no scattering of sound would appear, since there would be complete cancellation of the wavelets emitted from different parts of the medium. In a random distribution, however, statistical fluctuations of density occur, and these may scatter energy. The analysis is exactly analogous to the scattering of light by the atmosphere. Following Fowler<sup>17</sup>, one may calculate the density fluctuations to be expected in a given volume, and determine the scattering from these. More simply one may compute a time average of the radiation scattered from a volume large compared to the wave length. As the different bubbles move about, the relative phases of their scattered wavelets will vary, and as a result constructive and destructive interference will be equally likely. The total scattered intensity is, therefore, the sum of the intensities of the individual waves; the total energy scattered per unit volume will be  $n\sigma_s I$ , exactly as in the case where the bubbles were widely spaced. The

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17. R. H. Fowler, Statistical Mechanics, (Cambridge U. Press, 1929), p. 154.

absorption of sound also follows the same formulae in both cases.

When the density of bubbles changes discontinuously, sound is reflected from the interface. This is simply the familiar case of reflection and refraction at the boundary of two homogeneous media, and may be treated by similar methods. The velocity of sound in bubbly water may be found either by the superposition of the direct and scattered sound waves, or by a consideration of the compressibility of the air-water mixture. The second method will be followed here because of its greater simplicity.

In general, we have for  $c$ , the velocity of sound in a medium

$$c^2 = \frac{dp}{d\rho} \quad , \quad (4-5)$$

which may be written

$$c^2 = \frac{dp/dt}{d\rho/dt} \quad . \quad (4-6)$$

The quantity  $p$  is given in equation (1-1). If the expansion and contraction of a unit volume is considered.

$$\frac{dp}{dt} = -p \frac{dV}{dt} \quad , \quad (4-7)$$

where  $V$  is the sum of  $V_e$ , the volume of water, or any liquid external to the bubbles, and  $V_1$ , the volume of the bubbles. The average value of  $V_1$  is the relative amount of air present, by volume, in the air-water mixture; this quantity, which will be

denoted by  $u$ , is given by the relationship

$$u = n \cdot \frac{4\pi R^3}{3} \quad (4-8)$$

where  $n$  is the number of bubbles per  $\text{cm}^3$ . The quantity  $\rho$  will be set equal to the density of the water, and  $u$  will be neglected compared to unity. The results should be valid for  $u < .03$ , corresponding to a distance between bubbles of roughly  $5R$ .

If equations (1-14) and (1-20) are used, equation (4-6) gives

$$c^2 = \frac{-i\omega p_0 e^{i\omega t}}{\rho \left\{ \frac{dV_e}{dt} - 4\pi n A e^{i\omega t} \right\}} \quad (4-9)$$

The constant  $A$  is given by equation (2-18); since also  $c$  must equal  $c_0$  when  $n$  vanishes, we have

$$c^2 = \frac{c_0^2}{1 + \frac{4\pi R n / \rho^2}{(\omega_r^2 / \omega^2 - 1) + i\delta}} \quad (4-10)$$

Equation (4-10) may also be written in the form

$$c^2 = \frac{c_0^2}{1 + \frac{3u/a^2}{(\omega_r^2/a\omega^2 - 1) + i\delta}} \quad (4-11)$$

Then quantity  $a$  is equal to  $kR$ , or  $2\pi R/\lambda$ ; the value of  $a$  at adiabatic resonance, denoted by  $a_0$ , is given in equation (1-36).

For frequencies far from resonance, the imaginary term in equation (4-11) may be neglected. For long wave-length radiation,  $a_o$  is much greater than  $a$ , and

$$c = \frac{c_o}{(1 + 3\alpha u/ga_o^2)^{1/2}} \quad (4-12)$$

This result is independent of the bubble radius. Even for  $10^{-4}$  parts of air at atmospheric pressure per 1 part of water,  $c$  will be reduced to 62 percent of its value with no air when  $\alpha$  and  $g$  are unity. For air bubbles in water whose radii are less than  $10^{-4}$  cm, however,  $g$ , defined in equation (2-21), becomes important. Equation (4-12) is valid only for values of  $u$  small compared to unity. A curve which shows  $c$  for all values of  $u$  for bubbles below resonance has been given by Wood;<sup>18</sup> his curve neglects the effect of surface tension, however.

For sound of short wave-length,  $a_o/a$  is negligible and

$$c = \frac{c_o}{(1 - 3u/a^2)^{1/2}} \quad (4-13)$$

Equation (4-13) gives the perhaps unexpected result that when  $u$  is equal to  $a^2/3$ ,  $c$  is infinite, and for a greater amount of air, no sound waves can be present in the medium. This effect may be traced back to the fact that a bubble bigger than resonant size has its greatest radius when the pressure is greatest; when there

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18. A. B. Wood, A Textbook of Sound (Macmillan, 1941), p. 362.

are enough of such bubbles their expansion more than offsets the compression of the water, and  $dp/d\rho$  is negative. For bubbles of twice the resonant size, with  $\alpha$  and  $g$  equal to unity, the critical value of  $u$  for which  $c$  is infinite is roughly  $2 \times 10^{-4}$ , corresponding to a distance between the bubbles some 30 times the bubble radius. When  $u$  becomes comparable with unity, equation (4-13) is no longer valid, since changes in density become important, and in addition the amplitude of oscillation is no longer given by the analysis in Section II.

For normal incidence, the reflection coefficient  $r$  at an interface between two media of equal  $\rho$  but different  $c$  is equal to

$$r = \left( \frac{c_1 - c_2}{c_1 + c_2} \right)^2 \quad (4-14)$$

Near resonance, however, equation (4-14) cannot be used. If we write equation (4-10) in the form

$$\frac{c_o^2}{c} = a + ib, \quad (4-15)$$

and solve for  $\varphi$  and  $\zeta$ , the real and imaginary parts of  $c_o/c$ , then

$$\varphi^2 = \frac{a + \sqrt{a^2 + b^2}}{2}, \quad (4-16)$$

$$\zeta^2 = \frac{-a + \sqrt{a^2 + b^2}}{2}. \quad (4-17)$$

The imaginary part of  $c_0/c$  may be used to derive equation (4-1). The reflection coefficient  $r$  for normal incidence\* is given by

$$r = \frac{(1-\epsilon)^2 + \delta^2}{(1+\epsilon)^2 + \delta^2} \quad (4-18)$$

Equations (4-11) and (4-15) through (4-18) have been used to compute values of  $r$  as a function of  $\omega/\omega_0$  for different values of  $u$ , the volume of air per unit volume of air-water mixture. The results are plotted in Figure 7. The values of  $\alpha$  have been computed for a frequency of 24 kc; for other frequencies of practical interest the curves will not be very different. The theoretical values of  $\delta$  used in plotting Figures 3-5 have been used to compute the values of  $r$  in Figure 7.

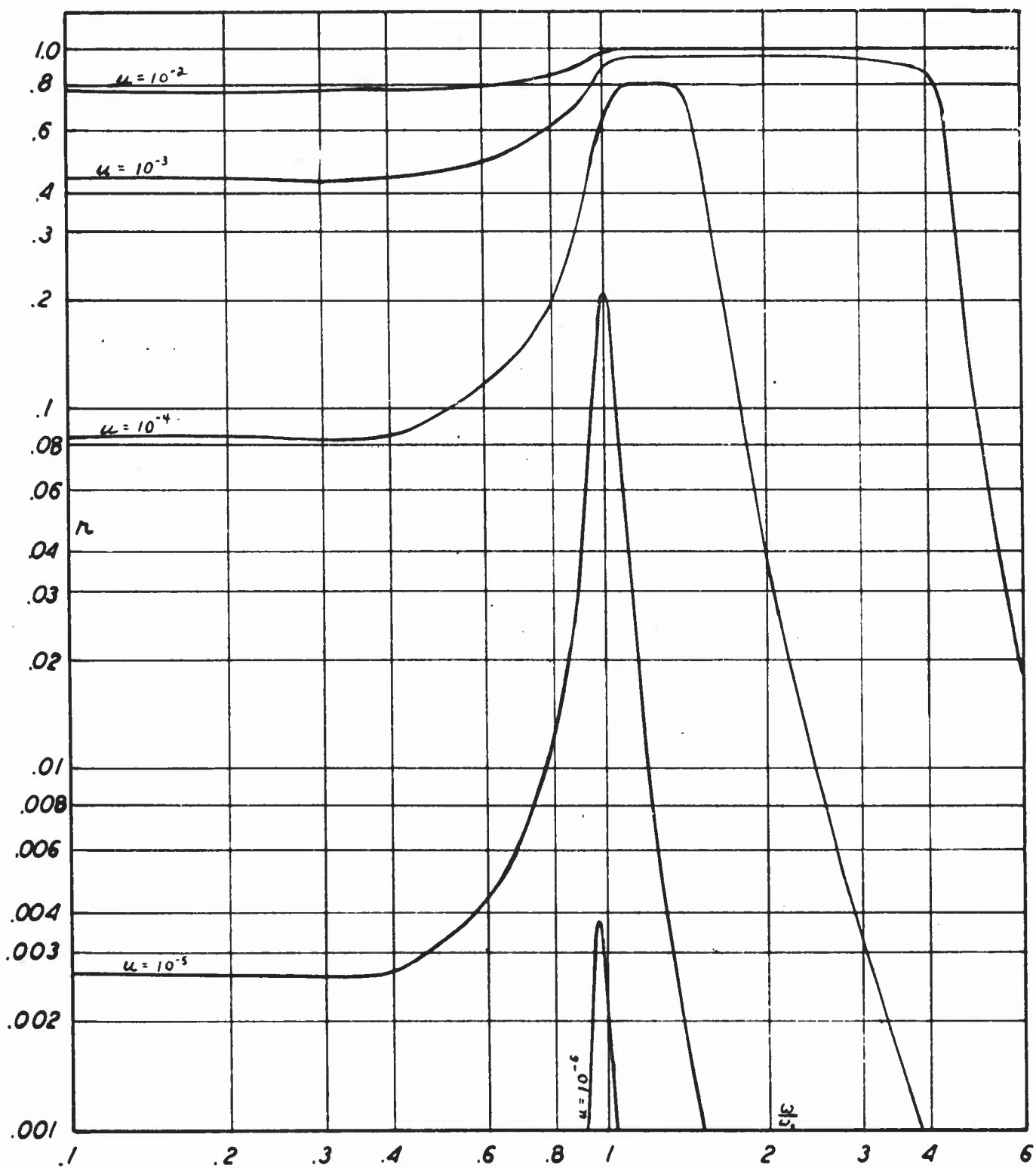
The values of  $r$  for very small  $u$  are of doubtful significance, especially for the larger bubbles, since the average spacing between bubbles becomes comparable with the wave length. In such cases, however, the reflected energy is small compared to the scattered energy. In fact the energy scattered from a layer of thickness  $x$  will be equal to the reflected energy when  $n\sigma_s x$  equals  $r$ , or when

$$x = \lambda n \left( \frac{\lambda}{4\pi} \right)^3, \quad (4-19)$$

for frequencies far from resonance. Thus if there is only one

\* $r$  is the fraction of the incident energy which is reflected.

FIGURE 7  
REFLECTION COEFFICIENTS



bubble, on the average, in a cube of length  $\lambda/4\pi$ , the scattering from a layer of thickness  $\lambda$  will just equal the reflection found from equation (4-14). It is only for very small bubbles, or great bubble densities, that the reflected energy determined from equation (4-18) is important. The values of  $r$  for large  $u$  are perhaps not precise, owing to the effects described above, but it is clear that the reflection coefficient is close to unity when  $u$  is as great as  $10^{-2}$ .

It should be noted that if bubbles of all sizes are present, with radii both less and greater than the radius of resonance, the change of the sound velocity will be less than computed from equation (4-11). In addition, if the changes in bubble density are gradual, and extend over several wave lengths, the reflection will be substantially less than that computed from equation (4-18).



# V. MANY BUBBLES OF DIFFERENT SIZES.

When bubbles are present in abundance in water, there will usually be a wide dispersion of bubble sizes. It is, therefore, of interest to determine the acoustical effects produced by such a distribution.

The effect produced by resonant bubbles is enormously greater than that arising from bubbles of other sizes. For most purposes the total reflection, scattering, and absorption of sound will be obtained by integrating only over the resonance peak in each case. This procedure will be followed first for the absorption and scattering and then for the reflection produced by closely spaced bubbles.

Let the number of bubbles per  $\text{cm}^3$  with a radius between  $R$  and  $R + dR$  be denoted by  $n(R) \cdot dR$ . If  $S_s$  and  $S_e$  denote the total scattering and extinction cross-sections per cubic centimeter, then from equation (2-17) and (2-18) we have

$$S_s = \int \frac{n(R) \cdot 4\pi R^2 dR}{\left(\frac{\omega R}{\omega_0} - 1\right)^2 + \delta^2} , \quad (5-1)$$

$$S_e = \int \frac{n(R) \cdot 4\pi R^2 \delta dR/a}{\left(\frac{\omega R}{\omega_0} - 1\right)^2 + \delta^2} . \quad (5-2)$$

Unless  $n(R)$  is very much less for resonant bubbles than for those

of other sizes, the integrals will depend primarily on a narrow range of integration over the resonance peak. We may, therefore, take outside of the integral all the quantities which vary slowly in this region, giving them their value at resonance. Also, the quantity  $\delta$  in the denominator may be given its value at resonance. A subscript r will again be used to denote the value of any quantity at resonance.

If we also make the substitution

$$\frac{\omega - \omega_r}{\omega_r} = \frac{R - R_r}{R_r} = \omega \quad ,$$

equations (5-1) and (5-2) yield

$$S_s = 4\pi R_r^3 n(R_r) \int_{-\infty}^{+\infty} \frac{d\omega}{4\omega^2 + \delta_r^2} \quad , \quad (5-3)$$

$$S_e = \frac{4\pi R_r^3 \delta_r n(R_r)}{a_r} \int_{-\infty}^{+\infty} \frac{d\omega}{4\omega^2 + \delta_r^2} \quad ; \quad (5-4)$$

these integrals have been extended to infinity for simplicity; half of their value arises from the range from  $-\delta_r$  to  $+\delta_r$ , and the rest comes from values of  $\omega$  which are not much greater. The integrals are equal to  $\pi/2\delta_r$ , and with this substitution  $S_s$  and  $S_e$  become

$$S_s = \frac{2\pi^2 R_r^3 n(R_r)}{\delta_r} \quad , \quad (5-5)$$

$$S_e = \frac{2\pi^2 R_r^3 n(R_r)}{a_r} . \quad (5-6)$$

It is interesting to note that the total extinction cross-section is independent of  $\delta_r$ . The absorption cross-section is simply

$$S_a = S_e - S_s . \quad (5-7)$$

In a cubic centimeter of the water-air mixture, let  $u(R)dR$  be the total volume of air contributed by bubbles with radii between  $R$  and  $R + dR$ . Then equations (5-5) and (5-6) yield

$$S_s = \frac{3\pi u(R_r)}{2\delta_r} , \quad (5-8)$$

$$S_e = \frac{3\pi u(R_r)}{2a_r} . \quad (5-9)$$

Let us define  $u_r$  as equal to  $R_r u(R_r)$ . If  $u(R)$  were constant from  $R = 0.5R_r$  to  $R = 1.5R_r$ , and zero outside this range,  $u_r$  would be the total volume of air per unit volume of the air-water mixture. The quantity  $u_r$  is the volume of resonant bubbles per  $\text{cm}^3$  per unit interval of  $\log R$ . Equations (5-8) and (5-9) yield, finally

$$S_s = \frac{3\pi u_r}{2\delta_r R_r} , \quad (5-10)$$

$$S_e = \frac{3\pi u_r}{2a_r R_r} . \quad (5-11)$$

If the results of Meyer and Tamm<sup>14</sup> are used,  $\delta_r R_r$  is roughly constant for frequencies from 5 to 25 kc, and equal to  $3.6 \times 10^{-3}$ ;  $a_r$ , on the other hand, equals  $1.36 \times 10^{-2} (g/\alpha)^{1/2}$ .

When sound is penetrating a medium with a particular value of  $S_e$ , the intensity will decay exponentially at the rate  $\exp(-S_e x)$ . For most purposes it is convenient to express this attenuation in terms of decibels per yard, a quantity which will be denoted by  $K_e$ ;  $K_s$  will be used to denote that part of the attenuation which arises from scattering. Numerically equations (5-10) and (5-11) yield

$$K_s = 5.2 \times 10^5 u_r, \quad (5-12)$$

$$K_e = \frac{1.4 \times 10^5}{R_r} \left( \frac{\alpha}{g} \right)^{1/2} u_r. \quad (5-13)$$

At 24 kc,  $R_r$  equals  $1.4 \times 10^{-2}$  cm,  $\alpha$  and  $g$  equal unity, and  $K_e$  is  $1.0 \times 10^7 u_r$ . If the theoretical formula for  $\delta_r$  is taken,  $K_s$  is reduced by a factor varying from about 2/3 at 5 kc to 1/3 at 30 kc. Equation (5-13) for  $K_e$  is independent of any assumption about damping constants, however, and should be accurate.

While for the calculation of attenuation, it is useful to express  $K_e$  directly in terms of db per yard, the scattered sound is most conveniently found from  $S_s$ , the scattering coefficient per unit volume. This same quantity has also been denoted<sup>19</sup>

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19. San Diego Reverberation Group, Reverberation at 24 kc, NDRC Report No. C4-sr30-401, November 23, 1942. The values of  $m$  given by the San Diego group are in terms of feet rather than yards. In equation (5-14) the right hand side must be divided by 3 if a comparison is to be made with the numerical values given in the San Diego report.

by  $m$ , and called the "volume scattering coefficient". Since  $e^{-S_s}$  is equal to  $10^{-K_s/10}$ ,  $S_s$  is equal to  $K_s(\log_e 10)/10$ , and from equation (5-12) we have

$$m = S_s = 1.2 \times 10^5 u_r \text{ yard}^{-1} \quad (5-14)$$

For some purposes it may be useful to know the average value of  $c_s/\pi R^2$  and  $c_e/\pi R^2$ . This information makes it possible to pass from the total geometrical (or optical) cross-section in the line of sight to the total absorption and scattering of sound. The total geometrical cross-section  $S_g$  per unit volume depends very much on the assumed distribution of bubble sizes. If  $u(R)$  is assumed constant over the range from  $R_r/2$  to  $3R_r/2$ , then

$$S_g = \frac{3u_r}{4R_r} \log_e 3 \quad (5-15)$$

If  $n(R)$  is assumed constant over the same range,  $13/16$ , or  $0.81$ , replaces the numerical constant  $\frac{3}{4} \log_e 3$ , or  $0.82$ . If a much greater number of very small bubbles is assumed,  $S_g$  will be much increased for a given  $u_r$ . If equations (5-10), (5-11), and (5-15) are taken for  $S_s$ ,  $S_e$ , and  $S_g$ , the desired ratios become

$$\frac{S_s}{S_g} = \frac{2\pi}{\log_e 3} \times \frac{1}{\delta_r} = \frac{5.7}{\delta_r} \quad (5-16)$$

$$\frac{S_e}{S_g} = \frac{2\pi}{\log_e 3} \times \frac{1}{a_o} = 4.2 \times 10^2 \quad (5-17)$$

provided that  $g$  and  $\alpha$  are equal to unity. Thus if the bubbles

present in a region have a total cross-section of one per unit area, corresponding to a diminution of a light beam by a factor  $1/e$  in passing through the region, the acoustical absorption will be  $4.2 \times 10^2$  times as great, giving a total attenuation of 30 db through the region. This result is valid, of course, only if the distribution of bubble sizes resembles at least approximately the distribution used in deriving equation (5-15).

While the attenuation  $K_e$  may be interpreted directly in terms of observed quantities,  $K_s$  and  $S_s$  are not always directly applicable to observational data, owing to the complexities arising from multiple scattering. In the general case, an exact solution depends on the equation of radiative transfer, which has been extensively studied in the astronomical literature.

In two special cases, however,  $S_s$  has a direct observational significance. On the one hand, if the total attenuation throughout the bubble-filled region is small, that is if  $K_e X$  is small, where  $X$  is the thickness of the region, then  $S_s X$  or  $mX$  will be the fraction of the incident radiation which is scattered.

On the other hand, if the scattering coefficient  $S_s$  is very much less than the corresponding attenuation or extinction coefficient  $S_e$  (and  $K_s$  is correspondingly less than  $K_e$ ) multiple scattering becomes unimportant in most situations, and the total scattered radiation may again be computed. In a layer such that  $K_e$  and  $K_s$  are constant, the incident sound at a distance  $x$  into the layer will equal  $I \exp(-S_e x)$ , where  $I$  is the intensity inci-

dent on the layer, and the sound scattered from each thickness  $dx$  will be  $S_s I \exp(-S_e x) dx$ . The sound coming back out of the layer will also be absorbed, and scattered, but if the absorption is very great compared to the scattering, the sound  $I_s$  scattered straight back will be, per unit solid angle

$$I_s = \int S_s \frac{I e^{-2S_e x}}{4\pi} dx, \quad (5-18)$$

integrated over the thickness of the layer. The factor 2 in the exponent represents the fact that the sound is absorbed on the way back out as well as on the way in. If the total attenuation through the layer is large, then the integral may be taken from zero to infinity, and

$$I_s = \frac{S_s}{2S_e} \times \frac{I}{4\pi}; \quad (5-19)$$

hence the energy scattered directly backwards is equal to what would be found if a fraction  $K_s/4K_e$  of the incident sound energy were scattered uniformly in all backward directions; i.e., over a solid angle of  $2\pi$ . The ratio  $K_s/4K_e$  is equal to  $a_r/4J_r$ . Thus at 24 kc between 1 and 2 percent of the incident energy will be scattered backwards, if Meyer and Tamm's<sup>14</sup> results are used.

For comparison with equation (5-13) the absorption arising from bubbles other than those near resonance should be considered. In the general case the exact determination of  $S_s$  and  $S_a$  is dependent on the values of  $\sigma_s$  and  $\sigma_a$  for all bubble

sizes, and distributions can always be conceived for which bubbles other than resonance will be of primary importance.

For bubbles greater than resonance the effects are fairly simple. The scattering cross-section is simply equal to four times the geometrical or optical cross-section, and the absorption cross-section, as is evident from Figure 5, is quite small. Such bubbles may be the primary source of scattering if the relative number of resonant bubbles is very small, but are less likely to be the chief source of absorption.

Bubbles smaller than the resonant size, on the other hand, have an exceedingly small scattering cross-section but may contribute to the absorption. The scattering cross-section  $\sigma_s$  varies as the square of the volume when  $\omega/\omega_0$  is small, so that the larger bubbles have very much greater weight in determining  $S_s$  than do the smaller ones. The absorption cross-section of very small bubbles is very much greater than the scattering cross-section, however, and under some situations the absorption from such bubbles may be important.

There are two situations in which more detailed evaluation of the absorption is perhaps desirable, first for bubbles somewhat below the resonant size, secondly for very small microscopic bubbles. It is evident from Figure 5 that  $\sigma_a/\pi R^2$  at 6 and 24 kc is relatively constant with frequency for a certain range of  $\omega/\omega_0$  below resonance. This arises from the increase of  $\beta$  with decreasing frequency, which helps to offset the increase



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of the denominator in equation (2-27). This range of  $\omega/\omega_0$ , when  $\sigma_a/\pi R^2$  changes relatively slowly, has been called the "knee" of the absorption-frequency curve by Willis<sup>2</sup>, who attributes most of the absorption to this region, on the ground that the resonance peak, while high, is too narrow to contribute appreciably to the absorption.

Computation shows that this knee is somewhat less important than the resonance region, if the same optical cross-section per unit radius interval is assumed for the bubbles in each region. In Figure 5 the curve for 6 kc shows that  $\sigma_a/\pi R^2$  has a "knee" at a value of roughly 10, for  $\omega/\omega_0$  between 0.1 and 0.7. The resonance peak, on the other hand, shows a value of  $4 \times 10^3$  for  $\sigma_a/\pi R^2$ , 400 times the value at the "knee", while the relative half-width of the peak is about 0.04, roughly one-fifteenth as wide as the "knee"; the total contribution of the resonance peak is therefore 30 times that of the knee if the optical cross-section of the bubbles in an interval  $dR$ , or  $R^2 n(R)$ , is roughly the same in both regions. Roughly the same conclusion holds at 24 kc, since the resonant peak contributes the same amount as at 6 kc; the "knee", with roughly twice as great a value of  $\sigma_a/\pi R^2$  as at 6 kc, but extending over only one-half as great a range of  $\omega/\omega_0$ , also makes roughly the same contribution to  $S_e$  as before. One may conclude that for a given geometrical cross-section per unit radius interval, bubbles at the "knee" of the absorption-frequency curve contribute about 1/20 to 1/40 as much as the resonance peak.

For sufficiently small bubbles, viscosity may be more important than heat conduction. If surface tension were negligible so that  $g$  could be set equal to unity in equation (2-20) for  $\omega_r$ , then  $\sigma_a$  would be proportional to  $R^3$  when viscosity is dominant, and for a given frequency the attenuation would be proportional to the volume of air present, and independent of the distribution of bubble sizes. This was the result found by Epstein.<sup>10</sup>

For air bubbles in water at a pressure of one atmosphere,  $g$  cannot be neglected when viscosity is important, however, since in this case  $g$  equals 2 when  $R$  equals  $10^{-4}$  cm, one-third of the radius below which viscosity becomes important. For smaller values of  $R$ ,  $g$  is roughly proportional to  $1/R$ . The absorption cross-section  $\sigma_a$  is proportional to  $1/\omega_r^4$ , and since  $\omega_r^2$  varies as  $g$ , the presence of surface tension introduces an extra factor  $R^2$  into  $\sigma_a$ , and materially reduces the absorption produced by microscopic bubbles. If equations (1-30), (2-20), and (1-29) are substituted into equation (2-27), the total absorption cross-section per unit volume produced by very small bubbles becomes

$$S_a = \frac{4\mu c p \alpha^2 \omega^2}{3\gamma^2 P_0^2} \int \frac{u(R) dR}{g^2}, \quad (5-20)$$

integrated over the region in which the absorption is primarily the result of viscosity. It may be useful to express this result in terms of  $K_a$ , the absorption in db per yard. For air bubbles in water at atmospheric pressure, and for a frequency of 24 kc,  $\alpha$  equals  $\gamma$  and equation (5-20) may be written

$$K_a = \frac{1.8 \times 10^4 u}{\bar{g}^2}, \quad (5-21)$$

where the bar denotes a harmonic mean of  $g^2$ , and  $u$  is the total relative volume of the bubbles in question.

It should be pointed out that these formulae for microscopic bubbles must presumably be modified when bubbles are adhering to solid particles. While the amplitude of oscillation and the viscous dissipation are probably not much affected by the presence of a rigid surface, the influence of surface tension may be completely changed. For a bubble caught in a cup-like depression in a dust particle for instance, surface tension would have much less effect in increasing  $P_1$ , the average pressure inside the bubble, than it would for an isolated bubble; as a result  $g$  might be substantially less than the value found from equation (2-21). It is not impossible that if such small bubbles are maintained in this way, their contribution to  $K_a$  should be computed from equation (5-21) with  $\bar{g}^2$  set equal to unity.

The results developed so far in this Section are applicable to the scattering and absorption produced by a uniform distribution of bubbles, either closely or widely spaced. For high bubble densities, however, there are also reflection effects to be taken into account whenever the density of bubbles changes appreciably in a region small compared to the wave length. If the bubbles are predominantly smaller or larger than resonance,

the analysis in Section IV may be applied. In the more important case, however, the chief effect arises from bubbles in the neighborhood of resonance.

The change of velocity may as before be computed from the total change in volume in a unit cube of air-water mixture. In equation (4-10) the term in the denominator must be integrated over a distribution of bubble sizes. The integration over the resonant peak contributes nothing to the real part of this integral, since positive and negative values of the integral are equally likely if  $Rn(R)$  is essentially constant over the resonance peak. The imaginary part does not vanish, however, and it is readily shown that

$$\frac{c_o^2}{c^2} = 1 - \frac{3\pi i u_r}{2a_r^2} \quad , \quad (5-22)$$

where  $u_r$  equals  $Ru(R_r)$  as before.

The imaginary part of the wave velocity gives the same attenuation in db per kiloyard as was found in equation (5-13). If equations (4-15) to (4-18) are used to determine the reflection coefficient  $r$ , we have,

$$r = \frac{\sqrt{1 + \sqrt{1 + (2.5 \times 10^4 u_r)^2}} - \sqrt{2}}{\sqrt{1 + \sqrt{1 + (2.5 \times 10^4 u_r)^2}} + \sqrt{2}} \quad , \quad (5-23)$$

where the value of  $a_r$ , or  $a_o(g/\alpha)^{1/2}$ , at atmospheric pressure has been determined from equation (1-36), with  $\alpha$  and  $g$  both set

equal to unity. When  $u_r$  is  $10^{-5}$  or less, equation (5-23) becomes

$$r = (6.2 \times 10^3 u_r)^2 \quad . \quad (5-24)$$

When  $u_r$  is less than  $10^{-5}$ , the reflection coefficient given by this equation is small, and less than the scattered radiation, whose value then depends on the ratio  $a_r/4\delta_r$ . For greater values of  $u_r$ , a large fraction of the incident energy may be reflected even when  $K_s$  is much less than  $K_e$ .

Bubbles whose radii are far from the resonant value may also contribute to the reflection, of course. For bubbles whose radii are below the resonant radius  $R_r$  but greater than  $2T/P_0$  ( $10^{-4}$  cm for air bubbles in water at atmospheric pressure) the reflection coefficient is independent of  $R$ , as may be seen from Figure 7, and equations (4-12) and (4-14) may be applied directly to determine  $r$  in terms of  $u$ , the total volume of all such bubbles per  $\text{cm}^3$ . For bubbles of radii less than  $2T/P_0$ , or much greater than the resonant radii  $R_r$ , the denominator in equations (4-12) or (4-13) must be integrated over the different bubble sizes; and the resulting value of  $r$ , found by use of equation (4-15), will depend on the detailed distribution of bubble sizes.

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